WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR $p$-EVOLUTION EQUATIONS

ALESSIA ASCANELLI, CHIARA BOITI, AND LUISA ZANGHIRATI

Abstract. We consider $p$-evolution equations in $(t, x)$ with real characteristics. We give sufficient conditions for the well-posedness of the Cauchy problem in Sobolev spaces, in terms of decay estimates of the coefficients as the space variable $x \to \infty$.

1. Introduction and main result

Given an integer $p \geq 2$, we consider in $[0, T] \times \mathbb{R}$ the equation $Pu(t, x) = f(t, x)$, where $P$ is a differential operator of the form

\begin{equation}
P(t, x, D_t, D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t, x)D_x^j,
\end{equation}

with $D = \frac{i}{t} \partial_t$, $a_p \in C([0, T]; \mathbb{R})$ and $a_j \in C([0, T]; \mathcal{B}^\infty)$ for $0 \leq j \leq p - 1$, (here $\mathcal{B}^\infty = \mathcal{B}^\infty(\mathbb{R}_x)$ is the space of complex valued functions which are bounded on $\mathbb{R}_x$ together with all their $x$-derivatives). We are dealing with non-kovalewskian evolution operators; anisotropic evolution operators of the form (1.1) are usually called $p$-evolution operators.

The aim of this paper is to give sufficient conditions for $H^\infty$ well-posedness of the Cauchy problem

\begin{equation}
\begin{cases}
P(t, x, D_t, D_x)u(t, x) = f(t, x) & (t, x) \in [0, T] \times \mathbb{R} \\
u(0, x) = g(x) & x \in \mathbb{R}.
\end{cases}
\end{equation}

The condition that $a_p$ is real valued means that the principal symbol (in the sense of Petrowski) of $P$ has the real characteristic $\tau = -a_p(t)\xi^p$; by the Laz-Mizohata theorem (cf. [M]), this is a necessary condition to have a unique solution, in Sobolev spaces, of the Cauchy problem (1.2) in a neighborhood of $t = 0$, for any $p \geq 1$.

We immediately notice that the case $p = 1$ corresponds to a kovalewskian operator of hyperbolic type. For $p = 2$ the operator is of Schrödinger type, for $p = 3$ we have the same principal part as the Korteweg-De Vries equation.

Literature about well-posedness in Sobolev spaces of the Cauchy problem for hyperbolic operators is really wide; coming up to $p \geq 2$, many results of well-posedness in Sobolev spaces are available under the assumption that the coefficients $a_j$ of (1.1) are real (see, for instance, [A1], [A2], [AZ], [AC], [CC1], [CHR]). On the contrary, when the coefficients $a_j(t, x)$ for $1 \leq j \leq p - 1$ are not real, we only know results for $p = 2, 3$; all these results show that, in order to have a well-posed Cauchy problem in Sobolev spaces, a suitable decay in $x$ for the imaginary part of the coefficients is needed. To be more precise, in the case $p = 2$ the problem of giving necessary and/or sufficient conditions for $H^\infty$ well-posedness of (1.2) has been largely

2000 Mathematics Subject Classification. 35G10, 35A27.

Key words and phrases. $p$-evolution equations, $H^\infty$ well-posedness, pseudo-differential operators.
investigated (see, for instance, [I1], [I2], [B], [KB]); in particular, in [I1] Ichinose states that a necessary condition for $H^\infty$ well-posedness of the Cauchy problem for the operator
\begin{equation}
(1.3) \quad P = i\partial_t + \Delta + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + c(x),
\end{equation}
with $b_j, c \in B^\infty$, is the existence of non-negative constants $M$, $N$ such that for every $\varrho > 0$ the inequality
\begin{equation}
(1.4) \quad \sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}} \left| \sum_{j=1}^{n} \int_{0}^{\varrho} \Re b_j(x + \theta \omega)\omega_j d\theta \right| \leq M \log(1 + \varrho) + N
\end{equation}
holds. This condition is also sufficient (cf. [I2]) only in the case of space dimension $n = 1$. A slightly stronger sufficient condition is given in [KB]: the Cauchy problem (1.2) for $p = 2$ and $a_2(t) = -1/2$ is $H^\infty$ well posed if
\begin{equation}
(1.5) \quad \text{Im} a_1(t, x) = \mathcal{O}(\|x\|^{-\sigma}), \quad \sigma \geq 1, \quad \text{as } |x| \to \infty,
\end{equation}
uniformly with respect to $t \in [0, T]$. We explicitly notice that condition (1.5) for (1.1) is consistent with condition (1.4) for (1.3). This result has been generalized to the case $p = 3$ by Cicognani and Colombini in [CC2], where the authors prove that the Cauchy problem (1.2) is $H^\infty$ well posed if:
\begin{align*}
|\text{Im} a_2| &\leq C a_3(t) \langle x \rangle^{-1}, \\
|\text{Im} a_1| + |\text{Im} D_x a_2| &\leq C a_3(t) \langle x \rangle^{-1/2}.
\end{align*}

Well-posedness results in Sobolev spaces for higher order 2-evolution equations are also available (see, for instance, [D], [T], [ACC], [CR]).

In this paper we generalize the results of [KB] and [CC2] to the case $p \geq 4$, proving the following:

**Theorem 1.1.** Let us consider the operator (1.1) with the following assumptions on the coefficients:
\begin{align*}
a_p &\in C([0, T]; \mathbb{R}), \quad a_p(t) \geq 0 \ \forall t \in [0, T] \\
a_j &\in C([0, T]; B^\infty), \quad 0 \leq j \leq p - 1 \\
(1.6) \quad |\text{Re} D^\beta_x a_j(t, x)| &\leq C a_p(t) \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \quad 0 \leq \beta \leq j - 1, \ 3 \leq j \leq p - 1 \\
(1.7) \quad |\text{Im} D^\beta_x a_j(t, x)| &\leq \frac{C a_p(t)}{\langle x \rangle^{\beta/2 - 1}}, \forall (t, x) \in [0, T] \times \mathbb{R}, \quad 0 \leq \left\lfloor \frac{\beta}{2} \right\rfloor \leq j - 1, \ 3 \leq j \leq p - 1 \\
(1.8) \quad |\text{Im} a_2| &\leq \frac{C a_p(t)}{\langle x \rangle^{3/2 - 1}} \\
(1.9) \quad |\text{Im} a_1| + |\text{Im} D_x a_2| &\leq \frac{C a_p(t)}{\langle x \rangle^{1/2 - 1}}
\end{align*}
for some $C > 0$, where $\left\lfloor \beta/2 \right\rfloor$ denotes the integer part of $\beta/2$.

Then, the Cauchy problem (1.2) is well-posed in $H^\infty$ (with loss of derivatives). More precisely, there exists a positive constant $\sigma$ such that for all $f \in C([0, T]; H^s)$ and $g \in H^s$ there is a unique solution $u \in C([0, T]; H^{s-\sigma})$ which satisfies the following energy estimate:
\begin{equation}
(1.10) \quad \|u(t, \cdot)\|_{s-\sigma}^2 \leq C_s \left( \|g\|^2_s + \int_{0}^{t} \|f(\tau, \cdot)\|^2_s d\tau \right) \quad \forall t \in [0, T],
\end{equation}
for some $C_s > 0$. 
Remark 1.2. For $p = 2, 3$ conditions (1.6) and (1.7) are empty and assumptions (1.8), (1.9) coincide with those of [KB], [CC2].

Remark 1.3. The assumption that $a_p(t)$ is non-negative can be clearly substituted by the assumption that it is non-positive.

Remark 1.4. If there exists a positive constant $C$ such that $a_p(t) \geq C$ for every $t \in [0, T]$, then Levi-type conditions on the coefficients are not needed: we can put $C$ instead of $Ca_p(t)$ on the right hand-side of (1.6)-(1.9).

2. Preliminary results

In order to prove Theorem 1.1 by the energy method we write

\[ iP = \partial_t + ia_p(t)D_x^p + \sum_{j=0}^{p-1} ia_j(t, x) D_x^j \]

\[ = \partial_t + A(t, x, D_x) \]

and compute, for a solution $u(t, x)$ of (1.2),

\[
\frac{d}{dt} \|u\|_0^2 = 2 \Re \langle \partial_t u, u \rangle = 2 \Re \langle iPu, u \rangle - 2 \Re \langle Au, u \rangle
\]

\[ \leq \|f\|_0^2 + \|u\|_0^2 - 2 \Re \langle Au, u \rangle, \]

where $\|\cdot\|_0$ and $\langle \cdot, \cdot \rangle$ denote, respectively, the norm and the scalar product in $L^2(\mathbb{R})$.

We look for an estimate from below for $\Re \langle Au, u \rangle$ of the form

\[ \Re \langle Au, u \rangle \geq -c\|u\|_0^2 \]

for some $c > 0$. For this purpose we want to make use of the sharp-Gårding Theorem A.1 or the Fefferman-Phong inequality (A.3). We thus need to replace $A$ with an operator $A_\Lambda$ whose symbol $\sigma(A_\Lambda)$ has non-negative real part.

To this aim we construct $A_\Lambda := (e^{\Lambda})^{-1}A e^{\Lambda}$, with $e^{A(x, D_x)}$ a pseudo-differential operator of symbol $\Lambda(x, \xi)$ such that:

- $\Lambda(x, \xi)$ is real valued;
- $e^{\Lambda} \in S^\delta$, $\delta > 0$, so that $e^{\Lambda} : H^\infty \to H^\infty$;
- $e^{\Lambda}$ is invertible;
- $(e^{\Lambda})^{-1}$ has principal part $e^{-\Lambda}$;
- $\sigma(Re A_\Lambda)(t, x, \xi) \geq 0$.

Then we consider the Cauchy problem

\[
\begin{cases}
P_\Lambda v = f_\Lambda \\
v(0, x) = g_\Lambda
\end{cases}
\]

for $P_\Lambda := (e^{\Lambda})^{-1}Pe^{\Lambda}$, $f_\Lambda := (e^{\Lambda})^{-1}f$ and $g_\Lambda := (e^{\Lambda})^{-1}g$. Well-posedness of (2.2) in Sobolev spaces is clearly equivalent to that of (1.2) for $u(t, x) = e^{A(x, D_x)}v(t, x)$.

Following [CC2] and [KB] we construct the operator $\Lambda(x, D_x)$ by defining its symbol

\[
\Lambda(x, \xi) := \lambda_{p-1}(x, \xi) + \lambda_{p-2}(x, \xi) + \ldots + \lambda_1(x, \xi)
\]

with

\[
\lambda_{p-k}(x, \xi) := M_{p-k}\omega \left( \frac{\xi}{h} \right) \int_0^x \langle y \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle h} \right) dy \langle \xi \rangle h^{-k+1}, \quad 1 \leq k \leq p - 1,
\]

where $\omega$ is a function such that $\omega(0) = 1$.
where \( \langle \xi \rangle_h := \sqrt{h^2 + \xi^2} \) for \( h \geq 1 \), \( \langle y \rangle := \langle y \rangle_1 \), the constants \( M_{p-k} > 0 \) will be chosen in order to apply Theorems A.1 or A.3, \( \omega \in C^\infty(\mathbb{R}) \) and \( \psi \in C_0^\infty(\mathbb{R}) \) satisfy:

\[
\omega(y) = \begin{cases} 
0 & |y| \leq 1 \\
|y|^{p-1/y^{p-1}} & |y| \geq 2 
\end{cases}
\]

\[
0 \leq \psi(y) \leq 1 \quad \forall y \in \mathbb{R}
\]

\[
\psi(y) = \begin{cases} 
1 & |y| \leq \frac{1}{2} \\
0 & |y| \geq 1.
\end{cases}
\]

In the present section we give the properties of \( \Lambda \) which will be used in §3 to prove Theorem 1.1.

**Lemma 2.1.** There exist positive constants \( C, \delta \) and \( \delta_{\alpha,\beta} \), independent on \( h \), such that

\[
|\Lambda(x, \xi)| \leq C + \delta \log \langle \xi \rangle_h \quad (2.5)
\]

\[
|\partial_\alpha \xi^\beta_x \Lambda(x, \xi)| \leq \delta_{\alpha,\beta} \langle \xi \rangle_h^{-\alpha} \quad \forall \alpha \in \mathbb{N}, \quad \beta \in \mathbb{N} \setminus \{0\}. \quad (2.6)
\]

**Proof.** Let us first remark that \( \psi((y)/\langle \xi \rangle_h^{p-1}) \) is zero outside

\[
E_\psi := \{ y \in \mathbb{R} : \langle y \rangle \leq \langle \xi \rangle_h^{p-1} \}.
\]

We can thus estimate, for \( k = 1 \),

\[
|\lambda_{p-1}(x, \xi)| \leq M_{p-1} \int_{E_\psi} \frac{1}{\sqrt{1 + y^2}} dy \leq M_{p-1} \log 2 + M_{p-1} (p - 1) \log \langle \xi \rangle_h; \quad (2.7)
\]

for \( 2 \leq k \leq p - 1 \),

\[
|\lambda_{p-k}(x, \xi)| \leq M_{p-k} \int_0^{\langle x \rangle} (y)^{-\frac{p-k}{p-1}} \chi_{E_\psi}(x) \langle \xi \rangle_h^{-k+1} dy \leq M_{p-k} \frac{p-1}{k-1} (x)^{\frac{k-1}{p-1}} \langle \xi \rangle_h^{-k+1} \chi_{E_\psi}(x) \quad (2.8)
\]

where \( \chi_{E_\psi} \) is the characteristic function of \( E_\psi \).

By (2.8) we get also, for \( 2 \leq k \leq p - 1 \), both:

\[
|\lambda_{p-k}(x, \xi)| \leq M'_{p-k} \quad (2.9)
\]

\[
|\lambda_{p-k}(x, \xi)| \leq M'_{p-k} \langle x \rangle^{\frac{k-1}{p-1}} \langle \xi \rangle_h^{-k+1}
\]

for \( M'_{p-k} = M_{p-k} \frac{p-1}{k-1} \).

Since

\[
|\Lambda(x, \xi)| \leq |\lambda_{p-1}(x, \xi)| + \sum_{k=2}^{p-1} |\lambda_{p-k}(x, \xi)|,
\]

from (2.7) and (2.9) we get (2.5) for \( \delta = (p - 1)M_{p-1} \) and \( C = M_{p-1} \log 2 + \sum_{k=2}^{p-1} M'_{p-k} \).
In order to prove (2.6), let us first consider the case $\alpha = 0$. For $\beta \geq 1$ and $1 \leq k \leq p - 1$ (using also the Faà Di Bruno formula for the derivative of a composite function):

\[
(2.10) \quad \partial_x^\beta \Lambda_p - k(x, \xi) = M_{p-k}\omega \left( \frac{\xi}{h} \right) \partial_x^{\beta-1} \left[ (\langle x \rangle)^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \right] (\langle x \rangle)^{-k+1}
\]

\[
= M_{p-k}\omega \left( \frac{\xi}{h} \right) \sum_{\beta' = 0}^{\beta-1} \binom{\beta - 1}{\beta'} \partial_x^{\beta'} (\langle x \rangle)^{-\frac{p-k}{p-1}} \partial_x^{\beta' - \beta} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) (\langle x \rangle)^{-k+1}
\]

\[
= M_{p-k}\omega \left( \frac{\xi}{h} \right) \partial_x^{\beta-1} (\langle x \rangle)^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) (\langle x \rangle)^{-k+1}
\]

\[
+ M_{p-k}\omega \left( \frac{\xi}{h} \right) \sum_{\beta' = 0}^{\beta-2} \binom{\beta - 1}{\beta'} \partial_x^{\beta'} (\langle x \rangle)^{-\frac{p-k}{p-1}}
\]

\[
\cdot \sum_{r_1 + \ldots + r_q = \beta - 1 - \beta'} C_{q, \nu} (q) \left( \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right) \partial_x^{r_1} \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \ldots \partial_x^{r_q} \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} (\langle x \rangle)^{-k+1}
\]

\[
(2.11) \quad := M_{p-k}\omega \left( \frac{\xi}{h} \right) A_{k, \beta} + M_{p-k}\omega \left( \frac{\xi}{h} \right) B_{k, \beta}
\]

with $B_{k, \beta} \equiv 0$ if $\beta = 1$.

Since $|\partial_x^r (\langle x \rangle)^{-\frac{p-k}{p-1}}| \leq c (\langle x \rangle)^{-\frac{p-k}{p-1}}$ and, on $E_\psi$,

\[
\left| \partial_x^{r_1} \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \ldots \partial_x^{r_q} \frac{\langle x \rangle}{\langle \xi \rangle_h^{p-1}} \right| \leq c \frac{(\langle x \rangle)^{1-r_1}}{\langle \xi \rangle_h^{p-1}} \ldots \frac{(\langle x \rangle)^{1-r_q}}{\langle \xi \rangle_h^{p-1}} \leq c (\langle x \rangle)^{-(r_1 + \ldots + r_q)} = c (\langle x \rangle)^{-\beta + 1 + \beta'}
\]

for some $c > 0$, there exist positive constants $c_{k, \beta}$ and $C_{k, \beta}$ such that

\[
(2.12) |\partial_x^\beta \Lambda_p - k(x, \xi)| \leq c_{k, \beta} M_{p-k}(\langle x \rangle)^{-\frac{p-k}{p-1} - \beta + 1} (\langle \xi \rangle_h^{k+1} \chi_{E_\psi}(x) = C_{k, \beta} (\langle x \rangle)^{\frac{k+1}{p-1} - \beta} (\langle \xi \rangle_h^{-k+1} \chi_{E_\psi}(x)
\]

\[
\leq C_{k, \beta} (\langle x \rangle)^{-\beta} \leq C_{k, \beta} \quad \forall 1 \leq k \leq p - 1.
\]

Therefore

\[
|\partial_x^\beta \Lambda(x, \xi)| \leq \sum_{k=1}^{p-1} C_{k, \beta} (\langle x \rangle)^{\frac{k+1}{p-1} - \beta + 1} (\langle \xi \rangle_h^{-k+1} \chi_{E_\psi}(x)
\]

\[
= (\langle x \rangle)^{-\frac{p}{p-1} - \beta + 1} (\langle \xi \rangle_h^{k+1} \chi_{E_\psi}(x)
\]

\[
\leq C_{\beta} (\langle x \rangle)^{-\frac{p}{p-1} - \beta + 1} (\langle \xi \rangle_h^{k+1} \chi_{E_\psi}(x)
\]

\[
(2.13) \quad = C_{\beta} (\langle x \rangle)^{-\beta}
\]

for some $C_{\beta} > 0$. 

For the case \( \alpha \geq 1 \) and \( 1 \leq k \leq p - 1 \), let us compute (for \( \beta = 0 \)):

\[
\partial_{\xi}^{\alpha} \lambda_{p-k}(x, \xi) = M_{p-k} \partial_{\xi}^{\alpha} \left[ \omega \left( \frac{\xi}{h} \right) \int_{0}^{x} \langle y \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{-1}} \right) dy \langle \xi \rangle_{h}^{-k+1} \right]
\]

\[
= M_{p-k} \left[ \partial_{\xi}^{\alpha} \omega \left( \frac{\xi}{h} \right) \right] \int_{0}^{x} \langle y \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{-1}} \right) dy \langle \xi \rangle_{h}^{-k+1}
\]

\[
+ M_{p-k} \sum_{\alpha' = 1}^{\alpha} \left( \frac{\alpha'}{\alpha} \right) \partial_{\xi}^{\alpha-\alpha'} \omega \left( \frac{\xi}{h} \right) \partial_{\xi}^{\alpha'} \left[ \int_{0}^{x} \langle y \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{-1}} \right) dy \langle \xi \rangle_{h}^{-k+1} \right]
\]

(2.14) \[= M_{p-k} (D_{k,\alpha} + E_{k,\alpha}). \]

Note that \( \omega(\xi/h) \) is constant for \( |\xi| \geq 2h \), and hence

\[
\partial_{\xi}^{\alpha} \omega \left( \frac{\xi}{h} \right) = \frac{1}{h^{\gamma}} \omega^{(\gamma)} \left( \frac{\xi}{h} \right) \chi_{\{ |\xi| < 2h \}} \quad \forall \gamma \geq 1,
\]

where \( \chi_{\{ |\xi| < 2h \}} \) is the characteristic function of the set \( \{ \xi \in \mathbb{R} : |\xi| < 2h \} \). Therefore, on the support of \( \omega^{(\gamma)} \):

(2.15) \[
\left| \partial_{\xi}^{\alpha} \omega \left( \frac{\xi}{h} \right) \right| \leq C_{\gamma} \langle \xi \rangle_{h}^{-\gamma} \quad \forall \gamma \geq 0
\]

for some \( C_{\gamma} > 0 \).

From (2.7) and (2.8) it follows that

(2.16) \[|D_{1,\alpha}| \leq C_{1,\alpha} \langle \xi \rangle_{h}^{-\alpha} (1 + \log \langle \xi \rangle_{h}) \chi_{\{ |\xi| < 2h \}} \]

(2.17) \[|D_{k,\alpha}| \leq C_{k,\alpha} \langle x \rangle^{\frac{k-1}{p-1}} \langle \xi \rangle_{h}^{-\alpha-k+1} \chi_{E_{\phi}}(x) \quad \forall 2 \leq k \leq p - 1
\]

for some \( C_{1,\alpha}, C_{2,\alpha}, \ldots, C_{p-1,\alpha} \geq 0 \).

In order to estimate \( E_{k,\alpha} \) we consider, for \( \alpha' \geq 1 \):

(2.18) \[
\partial_{\xi}^{\alpha'} \left[ \int_{0}^{x} \langle y \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{-1}} \right) dy \langle \xi \rangle_{h}^{-k+1} \right] = \int_{0}^{x} \langle y \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{-1}} \right) dy \partial_{\xi}^{\alpha'} \langle \xi \rangle_{h}^{-k+1}
\]

\[
+ \sum_{\alpha', \alpha'' = 1}^{\alpha'} \left( \frac{\alpha'}{\alpha''} \right) \int_{0}^{x} \langle y \rangle^{-\frac{p-k}{p-1}} \partial_{\xi}^{\alpha''} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{-1}} \right) dy \partial_{\xi}^{\alpha'-\alpha''} \langle \xi \rangle_{h}^{-k+1}
\]

\[
= F_{k,\alpha'} + G_{k,\alpha'}
\]

Note that \( F_{1,\alpha'} = 0 \) since \( \alpha' \geq 1 \). Moreover, from (2.8):

(2.19) \[|F_{k,\alpha'}| \leq C_{k,\alpha'} \langle x \rangle^{\frac{k-1}{p-1}} \langle \xi \rangle_{h}^{-\alpha'-k+1} \chi_{E_{\phi}}(x) \quad \forall 2 \leq k \leq p - 1
\]

for some \( C_{2,\alpha'}, \ldots, C_{p-1,\alpha'} > 0 \).
Since $\alpha'' \geq 1$ in $G_{k,\alpha'}$, we have, for $1 \leq k \leq p - 1$:

\begin{equation}
|G_{k,\alpha'}| \leq \sum_{\alpha''=1}^{\alpha'} \left( \frac{\alpha'}{\alpha''} \right) \left| \int_0^\infty \langle y \rangle^{-\frac{p-k}{p-1}} \sum_{\substack{r_1+\ldots+r_q=\alpha'' \\ r_i \in \mathbb{N} \setminus \{0\}}} C_{q,r} \psi(q) \left( \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{p-1}} \right) \partial^{r_i} \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{p-1}} \ldots \partial^{r_q} \frac{\langle y \rangle}{\langle \xi \rangle_{h}^{p-1}} dy \right| \cdot \\
\cdot |\partial^{\alpha'-\alpha''}(\langle \xi \rangle_{h}^{-k+1})| \\
\leq \sum_{\alpha''=1}^{\alpha'} \left( \frac{\alpha'}{\alpha''} \right) \left| \int_0^\infty \langle y \rangle^{-\frac{p-k}{p-1}} \sum_{\substack{r_1+\ldots+r_q=\alpha'' \\ r_i \in \mathbb{N} \setminus \{0\}}} C_{q,r} \psi(q) \cdot \chi_{\text{supp } \psi}(y) \right| \cdot \\
\cdot \sum_{\substack{r_1+\ldots+r_q=\alpha'' \\ r_i \in \mathbb{N} \setminus \{0\}}} C_{q,r} \sup_{\mathbb{R}} \psi(q) \cdot \chi_{\text{supp } \psi}(x) \langle \xi \rangle_{h}^{-(r_1+\ldots+r_q)} \langle \xi \rangle_{h}^{-k+1-\alpha'+\alpha''} \\
\leq C_{\alpha'} \langle x \rangle^{\frac{k-1}{p-1}} \langle \xi \rangle_{h}^{-k+1-\alpha'} \chi_{\text{supp } \psi}(x) \\
\tag{2.20}
\end{equation}

for some $C_{\alpha'} > 0$, where $\chi_{\text{supp } \psi} \subseteq \{ x \in \mathbb{R} : \frac{1}{2} \langle \xi \rangle_{h}^{p-1} \leq \langle x \rangle \leq \langle \xi \rangle_{h}^{p-1} \}$ is the characteristic function of the support of $\psi'(\langle x \rangle/\langle \xi \rangle_{h}^{p-1})$.

From (2.14), (2.15), (2.18), (2.19) and (2.20) it follows that

\begin{equation}
|E_{k,\alpha}| \leq \sum_{\alpha''=1}^{\alpha} C_{\alpha,\alpha'} \langle \xi \rangle_{h}^{-\alpha+\alpha'} \langle x \rangle^{\frac{k-1}{p-1}} \langle \xi \rangle_{h}^{-\alpha'-k+1} \chi_{E_0}(x) = C_{\alpha}' \langle x \rangle^{\frac{k-1}{p-1}} \langle \xi \rangle_{h}^{-\alpha-k+1} \chi_{E_0}(x) \\
\tag{2.21}
\end{equation}

for some $C_{\alpha,\alpha'}, C_{\alpha}' > 0$.

Therefore, (2.14), (2.16), (2.17) and (2.21) give, for $\alpha \geq 1$:

\begin{align}
|\partial^{\alpha} \lambda_{p-1}(x, \xi)| & \leq C_{\alpha} M_{p-1} \langle \xi \rangle_{h}^{-\alpha} (1 + \log \langle \xi \rangle_{h} \chi(|\xi| < 2h)) \\
|\partial^{\alpha} \lambda_{p-k}(x, \xi)| & \leq C_{\alpha} M_{p-k} \langle x \rangle^{\frac{k-1}{p-1}} \langle \xi \rangle_{h}^{-\alpha-k+1} \chi_{E_0}(x) \\
& \leq C_{\alpha} M_{p-k} \langle \xi \rangle_{h}^{-\alpha} \quad \forall 2 \leq k \leq p - 1 \\
\tag{2.22}
\tag{2.23}
\end{align}

for some $C_{\alpha} > 0$. 

Let us finally assume $\alpha, \beta \geq 1$ and compute, from (2.10) and (2.11):

$$
\partial_\xi^\alpha \partial_\xi^\beta \lambda_{p-k}(x, \xi) = M_{p-k} \partial_\xi^\alpha \omega \left( \frac{\xi}{\hbar} \right) \partial_\xi^{\alpha-1} \left[ \langle x \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle^{p-1}} \right) \right] \langle \xi \rangle^{h^{-k+1}} 
+ M_{p-k} \sum_{\alpha' = 1}^\alpha \left( \frac{\alpha}{\alpha'} \right) \partial_\xi^{\alpha-\alpha'} \omega \left( \frac{\xi}{\hbar} \right) \partial_\xi^{\alpha'-1} \partial_\xi^{\alpha'} \left[ \langle x \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle^{p-1}} \right) \right] \langle \xi \rangle^{h^{-k+1}} 
= M_{p-k} \partial_\xi^\alpha \omega \left( \frac{\xi}{\hbar} \right) (A_{k, \beta} + B_{k, \beta}) 
+ M_{p-k} \sum_{\alpha' = 1}^\alpha \left( \frac{\alpha}{\alpha'} \right) \partial_\xi^{\alpha-\alpha'} \omega \left( \frac{\xi}{\hbar} \right) \partial_\xi^{\alpha'-1} \left[ \langle x \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle^{p-1}} \right) \right] \langle \xi \rangle^{h^{-k+1}} 
+ \langle x \rangle^{-\frac{p-k}{p-1}} \sum_{\alpha'' = 0}^{\alpha'} \left( \frac{\alpha'}{\alpha''} \right) \sum_{r_1+\ldots+r_q = \alpha''} C_{q,r} \psi(q) \left( \frac{\langle x \rangle}{\langle \xi \rangle^{p-1}} \right) \cdot \partial_\xi^{\alpha''} \langle \xi \rangle^{h^{-k+1}} 
= M_{p-k} \partial_\xi^\alpha \omega \left( \frac{\xi}{\hbar} \right) (A_{k, \beta} + B_{k, \beta}) 
+ M_{p-k} \sum_{\alpha' = 1}^\alpha \left( \frac{\alpha}{\alpha'} \right) \partial_\xi^{\alpha-\alpha'} \omega \left( \frac{\xi}{\hbar} \right) \partial_\xi^{\alpha'-1} \langle x \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle^{p-1}} \right) \partial_\xi^{\alpha'} \langle \xi \rangle^{h^{-k+1}} 
+ M_{p-k} \sum_{\alpha' = 1}^\alpha \left( \frac{\alpha}{\alpha'} \right) \partial_\xi^{\alpha-\alpha'} \omega \left( \frac{\xi}{\hbar} \right) \sum_{\beta' = 1}^{\beta-1} \left( \beta - 1 \right) \partial^{\beta-1-\beta'} \langle x \rangle^{-\frac{p-k}{p-1}} \cdot \partial_\xi^{\alpha'} \langle \xi \rangle^{h^{-k+1}} 
+ M_{p-k} \sum_{\alpha' = 1}^\alpha \left( \frac{\alpha}{\alpha'} \right) \partial_\xi^{\alpha-\alpha'} \omega \left( \frac{\xi}{\hbar} \right) \sum_{\beta' = 0}^{\beta-1} \left( \beta - 1 \right) \beta' \partial^{\beta-1-\beta'} \langle x \rangle^{-\frac{p-k}{p-1}} \cdot \partial_\xi^{\alpha'} \langle \xi \rangle^{h^{-k+1}} 
+ M_{p-k} \sum_{\alpha' = 1}^\alpha \left( \frac{\alpha}{\alpha'} \right) \partial_\xi^{\alpha-\alpha'} \omega \left( \frac{\xi}{\hbar} \right) \partial_\xi^{\alpha''} \langle x \rangle^{-\frac{p-k}{p-1}} \psi(q) \left( \frac{\langle x \rangle}{\langle \xi \rangle^{p-1}} \right) \cdot \partial_\xi^{\alpha'} \langle \xi \rangle^{h^{-k+1}}
$$

(2.24) \quad = A_{k, \alpha, \beta} + B_{k, \alpha, \beta} + C_{k, \alpha, \beta} + D_{k, \alpha, \beta}$.
Note that, for \( \beta' \geq 1 \),
\[
\left| \partial_x^{\beta'} \psi \left( \frac{(x)}{(\xi)_h^{p-1}} \right) \right| \leq \sum_{r_1 + \ldots + r_q = \beta'} \sum_{r_c \in \mathbb{N}\{0\}} C_{q,r} \left| \psi(q) \left( \frac{(x)}{(\xi)_h^{p-1}} \right) \partial_x^{r_1} x^{1-r_1} \partial_x^{r_q} x^{1-r_q} \right| \leq \sum_{r_1 + \ldots + r_q = \beta'} C_{q,r} \sup_\mathbb{R} \left| \psi(q) \chi_{\text{supp } \psi} \right| (x)^{1-r_1} \partial_x^{r_q} x^{1-r_q} \leq c(x)^{-\beta'}
\]
for some \( c > 0 \). Therefore, from (2.11), (2.12) and (2.15) we have, for \( 1 \leq k \leq p-1 \):
\[
|A_{k,\alpha,\beta}| + |B_{k,\alpha,\beta}| \leq C_{\alpha,\beta} \langle x \rangle^{\frac{1}{p-1}-\beta} \langle \xi \rangle^{-\alpha-k+1} \chi_{E_\psi}(x)
\]
\[
|C_{k,\alpha,\beta}| + |D_{k,\alpha,\beta}| \leq C_{\alpha,\beta} \langle x \rangle^{\frac{1}{p-1}-\beta} \langle \xi \rangle^{-\alpha-k+1} \chi_{\text{supp } \psi}(x)
\]
for some \( C_{\alpha,\beta} > 0 \).

We have thus proved the existence of some \( C_{\alpha,\beta} > 0 \) such that
\[
(2.25) \quad |\partial_x^\alpha \partial_\xi^\beta \Lambda_{p-k}(x, \xi)| \leq C_{\alpha,\beta} \langle x \rangle^{\frac{1}{p-1}-\beta} \langle \xi \rangle^{-\alpha-k+1} \chi_{E_\psi}(x)
\]
for \( \alpha \geq 1, \beta \geq 1, 1 \leq k \leq p-1 \).

Estimates (2.25) and (2.13) finally give (2.6), since \( \langle x \rangle \leq \langle \xi \rangle^{p-1} \) on \( E_\psi \). \( \square \)

**Remark 2.2.** The estimate (2.25) is more precise than the estimate (2.6) and will be useful in the sequel.

From Lemma 2.1 we get, following [CC2], Proposition 2.1:

**Lemma 2.3.** Let \( \Lambda(x, \xi) \) satisfy (2.5) and (2.6). Then the operator \( e^{\Lambda(x,D_x)} \), with symbol \( e^{\Lambda(x,\xi)} \in S^b \), is invertible and
\[
(2.26) \quad (e^\Lambda)^{-1} = e^{-\Lambda}(I + R)
\]
where \( I \) is the identity operator and \( R \) is of the form \( R = \sum_{n=1}^{+\infty} \Gamma_n \) with principal symbol \( r_{-1}(x, \xi) = \partial_\xi \Lambda(x, \xi) D_x \Lambda(x, \xi) \in S^{-1} \).

**Proof.** Let us compute the symbol of \( e^{\Lambda(x,D_x)} e^{-\Lambda(x,D_x)} \):
\[
\sigma \left( e^{\Lambda(x,D_x)} e^{-\Lambda(x,D_x)} \right) = \sum_{m \geq 0} \frac{1}{m!} \partial_\xi^m e^{\Lambda(x,\xi)} D_x^m e^{-\Lambda(x,\xi)}
\]
\[
= 1 - \partial_\xi \Lambda(x, \xi) D_x \Lambda(x, \xi) + \sum_{m \geq 2} \frac{1}{m!} \partial_\xi^m e^{\Lambda(x,\xi)} D_x^m e^{-\Lambda(x,\xi)}.
\]
By (2.6),
\[
r_{-1}(x, \xi) := \partial_\xi \Lambda(x, \xi) D_x \Lambda(x, \xi) \in S^{-1}
\]
and
\[
\sum_{m \geq 2} \frac{1}{m!} \partial_\xi^m e^{\Lambda(x,\xi)} D_x^m e^{-\Lambda(x,\xi)} \in S^{-2}.
\]
Setting
\[
(2.27) \quad r(x, \xi) := \partial_\xi \Lambda(x, \xi) D_x \Lambda(x, \xi) - \sum_{m \geq 2} \frac{1}{m!} \partial_\xi^m e^{\Lambda(x,\xi)} D_x^m e^{-\Lambda(x,\xi)},
\]
we get (2.26) since (2.6), (2.22) and (2.23) imply, for some $C_{\alpha,\beta} > 0$:

$$\left| \partial_{\xi}^{\alpha} D_{x}^{\beta} r(x, \xi) \right| \leq C_{\alpha,\beta}(1 + \log h)(\xi)_h^{1-\alpha} = C_{\alpha,\beta}(1 + \log h)(h^2 + \xi^2)^{-1/2}(\xi)_h^{-\alpha}$$

(2.28)

$$\leq C_{\alpha,\beta}(1 + \log h)h^{-1}(\xi)_h^{-\alpha} \quad \forall \alpha, \beta \geq 0.$$  

This makes, for $h$ large enough, $I - r$ invertible by Neumann series with inverse operator $\sum_{n=0}^{+\infty} r^n$.

Similar arguments hold also for $e^{-A}e^{A}$, so that $e^{-A}\sum_{n=0}^{+\infty} r^n$ is a right inverse and a left inverse operator for $e^{A}$.

Remark 2.4. Once $h \geq 1$ is fixed large enough to get (2.26), the estimate (2.22) reduces to

$$\left| \partial_{\xi}^{\alpha} \lambda_{p-1}(x, \xi) \right| \leq C_{\alpha}(\xi)_h^{-\alpha}$$

for some $C_{\alpha} > 0$ depending also on the fixed $h \geq 1$.

Moreover, for $|\xi| > 2h$, from (2.14)-(2.20) we can also write, for some $C_{\alpha} > 0$,

(2.29) $$\left| \partial_{\xi}^{\alpha} \lambda_{p-1}(x, \xi) \right| \leq C_{\alpha}(\xi)_h^{-\alpha} \chi_{\text{supp } \psi}(x)$$

and hence, putting together (2.29) and (2.23):

(2.30) $$\left| \partial_{\xi}^{\alpha} \lambda_{p-k}(x, \xi) \right| \leq C_{\alpha} M_{p-k}(x) \frac{k^{k-1}}{(k-1)!} (\xi)_h^{-\alpha-k+1} \chi_{E_p}(x) \quad \forall 1 \leq k \leq p - 1.$$  

Lemma 2.5. Let $\Lambda(x, \xi)$ satisfy (2.6) and $h \geq 1$ be fixed large enough to get (2.26). Then

(2.31) $$\left| \partial_{\xi}^{\alpha} e^{\pm \Lambda(x, \xi)} \right| \leq C_{\alpha}(\xi)_h^{\alpha} e^{\pm \Lambda(x, \xi)} \quad \forall \alpha \in \mathbb{N}$$

(2.32) $$\left| D_{x}^{\beta} e^{\pm \Lambda(x, \xi)} \right| \leq C_{\beta}(x)^{-\beta} e^{\pm \Lambda(x, \xi)} \quad \forall \beta \in \mathbb{N}.$$  

Proof. Let us first remark that (2.31) and (2.32) are trivial for $\alpha = 0$ and $\beta = 0$.

From (2.30) we have, for $\alpha \geq 1$:

$$\left| \partial_{\xi}^{\alpha} e^{\pm \Lambda(x, \xi)} \right| = \left| \sum_{r_1 + \ldots + r_q = \alpha} C_{q,r}(\partial_{\xi}^{r_1} \Lambda) \ldots (\partial_{\xi}^{r_q} \Lambda) e^{\pm \Lambda(x, \xi)} \right|$$

$$\leq \sum_{r_1 + \ldots + r_q = \alpha} C'_{q,r}(\xi)_h^{-r_1 - \ldots - r_q} e^{\pm \Lambda(x, \xi)}$$

$$= C_{\alpha}(\xi)_h^{-\alpha} e^{\pm \Lambda(x, \xi)}$$

for some constants $C_{q,r}, C'_{q,r}, C_{\alpha} > 0$.

Analogously, from (2.13) for $\beta \geq 1$ we get:

$$\left| D_{x}^{\beta} e^{\pm \Lambda(x, \xi)} \right| = \left| \sum_{r_1 + \ldots + r_q = \beta} C_{q,r}(\partial_{x}^{r_1} \Lambda) \ldots (\partial_{x}^{r_q} \Lambda) e^{\pm \Lambda(x, \xi)} \right|$$

$$\leq C_{\beta}(x)^{-\beta} e^{\pm \Lambda(x, \xi)}$$

for some constant $C_{\beta} > 0$.  

From Lemma 2.3 we have also the following:

Lemma 2.6. Let $A(x, \xi) = ia_{p}(t)\xi^{p} + \sum_{j=0}^{p-1} ia_{j}(t, x)\xi^{j}$, $\Lambda(x, \xi)$ satisfying (2.6), and $r(x, \xi)$ as in (2.27).

Then the operator

$$A_{\Lambda}(t, x, D_{x}) := (e^{\Lambda(x,D_{x})})^{-1} A(t, x, D_{x}) e^{\Lambda(x,D_{x})}$$
can be written as
\[ A_\Lambda(t, x, D_x) = e^{-\Lambda(t, D_x)}A(t, x, D_x)e^{\Lambda(t, D_x)} \]
\[ + \sum_{m=0}^{p-2} \frac{1}{m!} \sum_{n=1}^{p-1-m} e^{-\Lambda(t, D_x)} A_{n,m}(t, x, D_x) e^{\Lambda(t, D_x)} + A_0(t, x, D_x), \]
(2.33)
where \( A_0(t, x, D_x) \) has symbol \( A_0(t, x, \xi) \in S^0 \) and
\[ \sigma(A_{n,m}(t, x, D_x)) = \partial^{m,n}(x, \xi) D_x^{m} A(t, x, \xi) \in S^{p-m-\nu}. \]

Proof. Let \( R(x, \xi) = \sum_{n=1}^{+\infty} r^n(x, \xi) \) as in Lemma 2.3, so that, from (2.26):
\[ (e^\Lambda)^{-1} Ae^\Lambda = e^{-\Lambda} Ae^\Lambda + e^{-\Lambda} RA e^\Lambda. \]

Note that
\[ \sigma(RA) = \sum_{m \geq 0} \frac{1}{m!} (\partial^{m,n}(x) D_x^m A) \]
\[ = \sum_{m \geq 0} \frac{1}{m!} \partial^{m,n}_\xi \left( \sum_{n=1}^{+\infty} r^n \right) (D_x^m A) \]
\[ = \sum_{m \geq 0} \frac{1}{m!} \sum_{n=1}^{+\infty} (\partial^{m,n}_\xi r) (D_x^m A) \]
since the series \( \sum_{n=1}^{+\infty} r^n(x, \xi) \) is normally convergent because of (2.28).

Moreover, for \( m \geq 1 \):
\[ \partial^{m,n}_\xi r^n(x, \xi) = \sum_{s_1 + \ldots + s_q = m, \ s_i \in \mathbb{N} \setminus \{0\}} C_{q,s} r^{n-q} \partial^{s_1}_\xi r \ldots \partial^{s_q}_\xi r, \]
for some \( C_{q,s} > 0 \). This means that \( \partial^{m,n}_\xi r \) is a symbol of order \( -(n-q) + (-1-s_1) + \ldots + (-1-s_q) = -n + q - q - (s_1 + \ldots + s_q) = -n - m \) for all \( m \geq 1 \) (and also for \( m = 0 \)).

Since \( D_x^m A \in S^p \) (in fact, \( D_x^m A \in S^{p-1} \) if \( m \geq 1 \) since \( a_p = a_p(t) \)), it follows that
\( (\partial^{m,n}_\xi r) (D_x^m A) \in S^0 \) if \( p - n - m \leq 0 \), i.e. \( n + m \geq p \).

We can thus restrict to \( n + m \leq p - 1 \), i.e. \( n \leq p - 1 - m \) and \( m \leq p - 1 - n \leq p - 2 \), and write
\[ \sigma(RA) = \sum_{m=0}^{p-2} \frac{1}{m!} \sum_{n=1}^{p-1-m} (\partial^{m,n}_\xi r) (D_x^m A) + R_0 \]
with \( R_0 \in S^0 \).

There exist then operators \( A_{n,m}(t, x, D_x) \) of order \( p - n - m \) and \( A_0(t, x, D_x) \) of order 0 such that
\[ R(x, D_x) A(t, x, D_x) = \sum_{m=0}^{p-2} \frac{1}{m!} \sum_{n=1}^{p-1-m} A_{n,m}(t, x, D_x) + A_0(t, x, D_x) \]
with \( \sigma(A_{n,m}(t, x, D_x)) = \partial^{m,n}(x, \xi) D_x^m A(t, x, \xi) \).

Substituting in (2.35) we finally get (2.33). □

We shall need in the sequel also the following:
Lemma 2.7. If $\Lambda$ is defined by (2.3) and (2.4), then, for $m \geq 1$,

$$e^{-\Lambda}D_x^me^\Lambda = \sum_{s=0}^{p-2} f_{-s}(\lambda_{p-1}, \ldots, \lambda_{p-s-1}) + f_{-p+1}(\lambda_{p-1}, \ldots, \lambda_1)$$

for some $f_{-p+1} \in S^{-p+1}$ depending on $\lambda_{p-1}, \ldots, \lambda_1$ and $f_{-s} \in S^{-s}$ depending only on $\lambda_{p-1}, \ldots, \lambda_{p-s-1}$, and not on $\lambda_{p-s}, \ldots, \lambda_1$, such that

$$|\partial^2_x \partial^\beta f_{-s}| \leq C_{\alpha,\beta,s} \frac{(\xi)^{s-\alpha}}{\langle x \rangle^{\frac{s}{p-1}+\beta}} \quad \forall \alpha, \beta \geq 0,$$

for some $C_{\alpha,\beta,s} > 0$.

Proof. For $m \geq 1$, by the Faà Di Bruno formula:

$$e^{-\Lambda}D_x^me^\Lambda = \sum_{r_1 + \ldots + r_q = m \atop r_i \in \mathbb{N}\setminus\{0\}} C_{q,r}(\partial^{r_1}x \lambda_{p-1} + \ldots + \partial^{r_q}x \lambda_1) \cdot (\partial^{q_1}x \lambda_{p-s_1}) \ldots (\partial^{q_s}x \lambda_{p-s_q}).$$

From (2.12) we have, for some $c > 0$:

$$|(\partial^{r_1}x \lambda_{p-s_1}) \ldots (\partial^{q_s}x \lambda_{p-s_q})| \leq c \langle x \rangle^{-\frac{p}{p-1}r_1+1} \langle \xi \rangle^{1-s_1} \ldots \langle x \rangle^{-\frac{p}{p-1}q_2+1} \langle \xi \rangle^{1-s_q} = c \langle x \rangle^{-\frac{p}{p-1}q_1+\ldots+q_s} = c \langle x \rangle^{-m+q} \langle \xi \rangle^{q-(s_1+\ldots+s_q)}.$$  

(2.38)

Note that $0 \leq s := (s_1 + \ldots + s_q) - q \leq q(p-1) - q = qp - q(p-1) = q > 0$, so that $(\partial^{r_1}x \lambda_{p-s_1}) \ldots (\partial^{q_s}x \lambda_{p-s_q}) \in S^{-s}$ and we can write

$$e^{-\Lambda}D_x^me^\Lambda = \sum_{s=0}^{m(p-2)} f_{-s}(x, \xi)$$

for some $f_{-s} \in S^{-s}$. Since

$$s_1 + \ldots + s_q = q + s \quad \Rightarrow \quad 1 \leq s_j \leq s + 1$$

it follows that $f_{-s}$ depends only on $\lambda_{p-1}, \ldots, \lambda_{p-s-1}$ for $0 \leq s \leq p-2$, while it depends on all $\lambda_{p-1}, \ldots, \lambda_1$ for $p-1 \leq s \leq m(p-2)$. Denoting by

$$f_{-p+1}(\lambda_{p-1}, \ldots, \lambda_1) := \sum_{s=p-1}^{m(p-2)} f_{-s} \in S^{-p+1},$$

we obtain (2.36).

Moreover (2.38) implies (2.37) for $\alpha, \beta = 0$ since, for $s_1 + \ldots + s_q = q + s$,

$$\langle x \rangle^{-\frac{pq-(s_1+\ldots+s_q)}{p-1}} = \langle x \rangle^{-\frac{pq-q-s}{p-1}} \leq \langle x \rangle^{-\frac{m(p-1)-s}{p-1}} \leq \langle x \rangle^{-\frac{p+1-s}{p-1}}.$$

Analogously, looking at the construction of $f_{-s}$, from (2.25) and (2.30) it follows that

$$|\partial^\alpha_x \partial^\beta x f_{-s}| \leq C_{\alpha,\beta,s} \frac{(\xi)^{s-\alpha}}{\langle x \rangle^{\frac{s}{p-1}+\beta}}$$

for some $C_{\alpha,\beta,s} > 0$. The thesis is thus proved. \[\square\]
3. Proof of the Main Theorem 1.1

Let

\[ \Lambda(x, D_x) = \lambda_{p-1}(x, D_x) + \ldots + \lambda_1(x, D_x) \]

where each \( \lambda_{p-k}(x, D_x) \) has symbol \( \lambda_{p-k}(x, \xi) \) defined as in (2.4). Fix \( h \geq 1 \) large enough so that the Neumann series \( R = \sum_{n=1}^{+\infty} r^n \) in (2.26) converges. Set then

\[ A(t, x, D_x) = \sum_{j=0}^{p} i a_j(t, x) D_x^j \]

with \( a_p(t, x) = a_p(t) \).

Our goal is to prove that, for \( A_\Lambda = (e^{\Lambda})^{-1} A e^\Lambda \),

\[ \text{Re}(A_{\Lambda}v, v) \geq -c\|v\|_0^2 \quad \forall v(t, \cdot) \in H^\infty \]

for some \( c > 0 \).

The proof is divided into the following steps:

**Step 1.** We compute the symbol of the operator \( e^{-\Lambda} A e^\Lambda \) and show that its terms of order \( p - k \), \( 1 \leq k \leq p - 1 \), denoted by \( (e^{-\Lambda} A e^\Lambda)|_{\text{ord}(p-k)} \), have the “right decay at the right level”, in the sense that they satisfy

\[ \left| \text{Re}(e^{-\Lambda} A e^\Lambda)|_{\text{ord}(p-k)}(t, x, \xi) \right| \leq C(M_{p-1}, \ldots, M_{p-k}) a_p(t) |x|^{-\frac{p-k}{p-1}} |\xi|^{p-k} \]

for a positive constant \( C(M_{p-1}, \ldots, M_{p-k}) \) depending only on \( M_{p-1}, \ldots, M_{p-k} \) and not on \( M_{p-k-1}, \ldots, M_1 \). This will be very important in the following in the application of the sharp-Gårding Theorem, since we shall choose \( M_{p-1}, \ldots, M_1 \) step by step, and at each step (say “step \( p - k \)”) we need something which depends only on the already chosen \( M_{p-1}, \ldots, M_{p-k+1} \) and on the new \( M_{p-k} \) that we need to choose, and not on the constants \( M_{p-k-1}, \ldots, M_1 \) which will be chosen in the next steps.

**Step 2.** In Steps 2,3,4 we choose, recursively, positive constants \( M_{p-1}, \ldots, M_1 \) in such a way that

\[ \text{Re} (e^{-\Lambda} A e^\Lambda)|_{\text{ord}(p-k)} + \tilde{C} \geq 0 \]

for some \( \tilde{C} > 0 \).

Here we choose \( M_{p-1} > 0 \) such that (3.2) holds for \( k = 1 \) and apply the sharp-Gårding Theorem A.1 to \( (e^{-\Lambda} A e^\Lambda)|_{\text{ord}(p-1)} + \tilde{C} \) to get

\[ \sigma(e^{-\Lambda} A e^\Lambda) = i a_p \xi^p + Q_{p-1} + \sum_{k=2}^{p-1} (e^{-\Lambda} A e^\Lambda)|_{\text{ord}(p-k)} + R_{p-1} + A_0, \]

where \( A_0 \in S^0 \) and \( R_{p-1} \) is a remainder (of order \( p - 2 \)) coming from the application of the sharp-Gårding Theorem A.1.

**Step 3.** To iterate this process, applying the sharp-Gårding Theorem A.1 to terms of order \( p - 2 \), \( p - 3 \), and so on, up to order 3, we need to investigate the action of the sharp-Gårding Theorem to each term of the form

\[ (e^{-\Lambda} A e^\Lambda)|_{\text{ord}(p-k)} + S_{p-k}, \]

where \( S_{p-k} \) denotes terms of order \( p - k \) coming from remainders of previous applications of the sharp-Gårding Theorem A.1, for \( p - k \geq 3 \).
We show at this step that remainders are sums of terms with “the right decay at the right level”, in the sense of (3.1). Then we apply the sharp-Gårding Theorem A.1 to terms of order \( p - k \), up to order \( p - k = 3 \).

**Step 4.** In this step we apply the Fefferman-Phong inequality to terms of order \( p - k = 2 \) and the sharp-Gårding inequality (A.2) to terms of order \( p - k = 1 \), finally obtaining that

\[
\sigma(e^{-\Lambda}Ae^{\Lambda}) = i\alpha_p \xi^p + \sum_{s=1}^{p} Q_{p-s}
\]

with

\[
\begin{align*}
\text{Re}(Q_{p-s}v, v) &\geq 0 \quad \forall v(t, \cdot) \in H^{p-s}, \quad s = 1, \ldots, p - 3 \\
\text{Re}(Q_{p-s}v, v) &\geq -c\|v\|_0^2 \quad \forall v(t, \cdot) \in H^{p-s}, \quad s = p - 2, p - 1 \\
Q_0 &\in S^0.
\end{align*}
\]

**Step 5.** We finally look at the full operator \( A_\Lambda \) in (2.33) and prove that \( e^{-\Lambda}A^{n,m}e^{\Lambda} \) satisfies the same estimates (3.1) as \( e^{-\Lambda}Ae^{\Lambda} \). Thus, the results of Step 4 hold for the full operator \((e^{\Lambda})^{-1}Ae^{\Lambda}\) and not only for \( e^{-\Lambda}Ae^{\Lambda} \), i.e. there exists a constant \( c > 0 \) such that

\[
\text{Re}(A_\Lambda v, v) \geq -c\|v\|_0^2 \quad \forall v(t, \cdot) \in H^\infty.
\]

From this, the thesis follows by standard energy methods.

We now proceed to the proof of the above mentioned steps.

**Step 1.** We compute first

\[
\sigma(A(t, x, D_x)e^{\Lambda(x,D_x)}) = \sum_{m \geq 0} \frac{1}{m!} \xi^m \left( \sum_{j=0}^{p} i\alpha_j(t, x)\xi^j \right) D_x^m e^{\Lambda(x, \xi)}
\]

\[
= \sum_{m=0}^{p} \sum_{j=0}^{m} \left( \frac{j}{m} \right) i\alpha_j(t, x)\xi^j D_x^m e^{\Lambda(x, \xi)}.
\]

Then, for some \( A_0 \in S^0 \) we have:

\[
\begin{align*}
\sigma(e^{-\Lambda}Ae^{\Lambda}) &= \sum_{\alpha \geq 0} \frac{1}{\alpha!} \xi^\alpha D_x^\alpha \left( \sum_{m=0}^{p} \sum_{j=0}^{m} \left( \frac{j}{m} \right) i\alpha_j\xi^j D_x^m e^{\Lambda} \right) \\
&= \sum_{\alpha \geq 0} \sum_{m=0}^{p} \sum_{j=0}^{m} \left( \frac{j}{m} \right) \left( \frac{\alpha}{\beta} \right) iD_x^\beta a_j(D_x^m + \alpha - \beta) e^{\Lambda} \xi^{j-m} \\
&= \sum_{m=0}^{p-1} \sum_{j=m+1}^{p} \sum_{\alpha \geq 0} \left( \frac{j}{m} \right) \left( \frac{\alpha}{\beta} \right) iD_x^\beta a_j(D_x^m + \alpha - \beta) e^{\Lambda} \xi^{j-m} + A_0 \\
&= \sum_{m=0}^{p-1} \sum_{j=m+1}^{p} \left( \frac{j}{m} \right) (i\alpha_j)(D_x^m e^{\Lambda}) \xi^{j-m} \\
&\quad + \sum_{m=0}^{p-2} \sum_{j=m+2}^{p} \sum_{\alpha \geq 0} \left( \frac{j}{m} \right) \left( \frac{\alpha}{\beta} \right) iD_x^\beta a_j(D_x^m + \alpha - \beta) e^{\Lambda} \xi^{j-m} + A_0 \\
&= A_I + A_{II} + A_0.
\end{align*}
\] (3.3)
We consider first $A_I$, where $\alpha \geq 1$. In the case $m + \alpha - \beta \geq 1$, from (1.6), (1.7), (1.8), (1.9), (2.30) and (2.12) we get:

$$
\left| (\xi^e \mathcal{D}_x^\alpha e^{-\Lambda})(\mathcal{D}_x^{m+\alpha-\beta} e^{\Lambda})\mathcal{X}^{j-m} \right|
$$

$$
= \left| \partial_\xi^\alpha \prod_{k=1}^{p-1} e^{-\lambda_{p-k}} \cdot |D_{x}^{\beta} a_j| \cdot |D_{x}^{m+\alpha-\beta} \prod_{k'=1}^{p-1} e^{\alpha_{p-k'}}| \right| \xi^{j-m}
$$

$$
\leq c a_p \sum_{\alpha_1 + \ldots + \alpha_{p-1} = \alpha} \alpha! \prod_{k=1}^{p-1} |\beta^{\alpha_k} e^{-\lambda_{p-k}}| \cdot \sum_{\gamma_1 + \ldots + \gamma_{p-1} = \frac{m+\alpha-\beta}{\alpha}} \frac{(m+\alpha-\beta)!}{\gamma_1! \cdots \gamma_{p-1}!} \prod_{k'=1}^{p-1} |\partial_{\xi}^{\alpha_{k'}} e^{\gamma_{p-k'}}| \left( \xi \right)^{j-m}
$$

$$
= c a_p \sum_{\alpha_1 + \ldots + \alpha_{p-1} = \alpha} \alpha! \prod_{k=1}^{p-1} e^{-\Lambda} \left( \sum_{r_1 + \ldots + r_{q_k} = \alpha_k} \frac{C_{q,k} |\partial_{\xi}^{\alpha_1} \lambda_{p-k} \cdots |\partial_{\xi}^{\alpha_k} |\partial_{\xi}^{\gamma_{p-k}}|}{\gamma_1! \cdots \gamma_{p-1}!} \right) \xi^{j-m}
$$

$$
(3.4) \leq c' a_p \sum_{\alpha_1 + \ldots + \alpha_{p-1} = \alpha} \prod_{k,k'=1}^{p-1} M_{p-k}^{q_k} \frac{\langle x \rangle_{\alpha_k + q_k(k-1)}}{\langle x \rangle_{\alpha_k + q_k(k-1)} \langle x \rangle_{\alpha_{k'} + q_{k'}(k'-1)}} \cdot M_{p-k'}^{q_{k'}} \frac{\langle x \rangle_{\alpha_{k'} + q_{k'}(k'-1)}}{\langle x \rangle_{\alpha_{k'} + q_{k'}(k'-1)} \langle x \rangle_{\alpha_k + q_k(k-1)}} \langle x \rangle_{\alpha_k + q_k(k-1)} \langle x \rangle_{\alpha_{k'} + q_{k'}(k'-1)} \langle x \rangle_{\alpha_k + q_k(k-1)}
$$

for some $c, c' > 0$. Note that we used here conditions (1.7) only for $0 \leq \beta \leq \alpha \leq j - m - 1 \leq j - 1$. The conditions on the further $D_x^{\alpha}$ derivatives will be required in the following to estimate the remainders coming from the sharp-Gårding Theorem A.1 for $j \geq 3$ (see (3.23)).

Each term of (3.4) has order

$$
j - m - \alpha - \sum_{k=1}^{p-1} q_k(k-1) - \sum_{k'=1}^{p-1} p_{k'}(k'-1)
$$

and decay in $x$ of the form

$$
\langle x \rangle_{\alpha_k + q_k(k-1) + \alpha_{k'} + q_{k'}(k'-1)}^{p-1} \leq \langle x \rangle_{\alpha_k + q_k(k-1) + \alpha_{k'} + q_{k'}(k'-1)}^{p-1}
$$

since $-(p-1)(m+\alpha-\beta) \leq -j + m + \alpha$ for $m + \alpha - \beta \geq 1$.

Note also that

$$
j - m - \alpha - \sum_{k=1}^{p-1} q_k(k-1) - \sum_{k'=1}^{p-1} p_{k'}(k'-1) \leq p - k - 1
$$

and

$$
j - m - \alpha - \sum_{k=1}^{p-1} q_k(k-1) - \sum_{k'=1}^{p-1} p_{k'}(k'-1) \leq p - k' - 1,
$$

so that whenever $M_{p-k}$ or $M_{p-k'}$ appear in (3.4), then the order is at most $p - k - 1$ and $p - k' - 1$ respectively.
In the case \( m + \alpha - \beta = 0 \), by (1.7), (1.8), (1.9) and (2.30) we have, for all \( 0 \leq \beta \leq j - 1 \) with \( 1 \leq j \leq p - 1 \):

\[
\begin{align*}
&|\text{Re}[(\partial^\xi e^{-\Lambda})(iD_x^{\beta}a_j)e^{\Lambda}j^{j-m}]| \\
&\leq |\partial^\xi e^{-\Lambda}| \cdot |\text{Im} D_x^{\beta}a_j| e^{\Lambda}(\xi)^{j-m} \\
&\leq \sum_{\alpha_1 + \ldots + \alpha_{p-1} = \alpha} \frac{\alpha!}{\alpha_1! \ldots \alpha_{p-1}!} \cdot \prod_{k=1}^{p-1} \left( \sum_{\substack{r_1 + \ldots + r_{q_k} = \alpha_k \\
r_1, \ldots, r_{q_k} \geq 1}} C_{q,k} \left| \partial^{\xi_1} \lambda_{p-k} \right| \cdots \left| \partial^{\xi_{q_k}} \lambda_{p-k} \right| \right) \cdot \frac{C_{p-k}(\xi)^{j-m}}{\langle x \rangle^{j-m}} \\
&\leq C' a_p \sum_{\alpha_1 + \ldots + \alpha_{p-1} = \alpha} \prod_{k=1}^{p-1} \sum_{\substack{r_1 + \ldots + r_{q_k} = \alpha_k \\
r_1, \ldots, r_{q_k} \geq 1}} M_{q_k}(\xi)^{k-1} q_k \langle \xi \rangle^{k-1} \frac{1}{\langle x \rangle^{j-m}} \\
\end{align*}
\]

(3.5) for some \( C' > 0 \).

Each term of (3.5) is a symbol of order \( j - m - \alpha - \sum_{k=1}^{p-1} q_k(k - 1) \) and has decay in \( x \) of the form

\[
\langle x \rangle^{-j-m-\sum_{k=1}^{p-1} q_k(k-1)} \leq \langle x \rangle^{-j-m-\sum_{k=1}^{p-1} q_k(k-1)}
\]

since \( [\beta/2] \leq \beta \leq \alpha + m \).

Here again

\[
j - m - \alpha - \sum_{k=1}^{p-1} q_k(k - 1) \leq p - k - 1
\]

and hence \( M_{p-k} \) appears in (3.5) only when the order is at most \( p - k - 1 \).

Summing up, formulas (3.4) and (3.5) give that the terms of order \( p - k \) of \( A_{II} \), denoted by \( A_{II|\text{ord}(p-k)} \), satisfy:

\[
|\text{Re} A_{II|\text{ord}(p-k)}| \leq \frac{C_{p-k}(\xi)^{p-k}}{\langle x \rangle^{j-m}}
\]

(3.6) for some \( C > 0 \).

Moreover, \( \text{Re} A_{II|\text{ord}(p-k)} \) depends only on \( M_{p-1}, \ldots, M_{p-k+1} \) and not on \( M_{p-k}, \ldots, M_1 \).

We consider then

\[
A_I = \sum_{m=0}^{p-1} \sum_{j=m+1}^{p} \binom{j}{m} (ia_j)(e^{-\Lambda}D_x^{m+1}e^{\Lambda})\xi^{j-m} \\
= \sum_{k=0}^{p-1} \sum_{m=0}^{k} \binom{p-k+m}{m} (ia_{p-k+m})(e^{-\Lambda}D_x^{m+1}e^{\Lambda})\xi^{p-k} \\
\]

(3.7) \[ i a_{p-k} \xi^{p-k} + \sum_{k=1}^{p-1} (ia_{p-k} \xi^{p-k} + \sum_{m=1}^{k} \binom{p-k+m}{m} (ia_{p-k+m})(e^{-\Lambda}D_x^{m+1}e^{\Lambda})\xi^{p-k}) \]

Note that \( D_x \Lambda = D_x \lambda_{p-1} + D_x \lambda_{p-2} + \ldots + D_x \lambda_1 \) with \( D_x \lambda_{p-k} \xi^{p-1} \in S^{p-k} \) because of (2.12).

Moreover, from Lemma 2.7 it follows that there exist \( f_{-s} \in S^{-s} \), for \( 0 \leq s \leq p - 2 \), depending
only on $\lambda_{p-1}, \ldots, \lambda_{p-s-1}$, and $f_{p+1} \in S^{-p+1}$ such that, for $\tilde{f}_0 = a_{p-k+m} f_{p+1} \xi^{p-k} \in S^0$, 

$$a_{p-k+m} (e^{-\Lambda} D_x^m e^\Lambda) \xi^{p-k} = \sum_{s=0}^{p-2} f_{-s}(\lambda_{p-1}, \ldots, \lambda_{p-s-1}) a_{p-k+m} \xi^{p-k} + \tilde{f}_0,$$

and, from (2.37) for $0 \leq s \leq p - 2$,

$$|f_{-s} a_{p-k+m} \xi^{p-k}| \leq \frac{C_s a_p}{\langle x \rangle^{p-1-s}} \langle \xi^{p-k-s} \rangle \leq \frac{C_s a_p}{\langle x \rangle^{p-1}} \langle \xi \rangle^{p-k-s} \quad \forall k \geq 1$$

for some $C_s > 0$, because of (1.6)-(1.9) (with $\beta = 0$ and not using the assumption on the decay in $x$).

Rearranging the terms of the second addend of $A_I$ in (3.7) and putting together all terms of order $p - k$, we can thus write, because of (3.8), (3.9):

$$A_I = ia_p \xi^p + \sum_{k=1}^{p-1} (ia_p \xi^{p-k} + ipa_p D_x \lambda_{p-k} \xi^{p-1} + B_{2-k} \xi^{p-2} + B_{3-k} \xi^{p-3} + \ldots + B_0 \xi^{p-k}) + \tilde{B}_0,$$

for some $\tilde{B}_0 \in S^0$ and $B_{s-k} \xi^{p-s} \in S^{p-k}$ of the form

$$B_{s-k} = b_{s-k}(\lambda_{p-1}, \ldots, \lambda_{p-s-k-1}) \sum_{m=1}^{k} a_{p-s+m} \in S \quad s \leq \frac{1}{2}$$

with

$$|B_{s-k}| \leq \frac{C_{s-k} a_p}{\langle x \rangle^{s-k}} \langle \xi \rangle^{s-k}$$

for some $C_{s-k} > 0$.

Setting

$$A_{0-k}^0 := ia_{p-k} \xi^{p-k} + ipa_p D_x \lambda_{p-k} \xi^{p-1}$$

$$A_{1-k}^1 := B_{2-k} \xi^{p-2} + \ldots + B_0 \xi^{p-k}$$

we write

$$A_I = ia_p \xi^p + \sum_{k=1}^{p-1} (A_{p-k}^0 + A_{p-k}^1) + \tilde{B}_0.$$

Note that $A_{p-k}^0, A_{p-k}^1 \in S^{p-k}$ and

$$|\text{Re } A_{p-k}^0| + |A_{p-k}^1| \leq \frac{C_s a_p \langle \xi \rangle^{p-k}}{\langle x \rangle^{s-k}}$$

for some $C_s > 0$ because of (1.7), (2.12) and (3.11).

Moreover, $A_{p-k}^0$ depends only on $M_{p-k}$ and $A_{p-k}^1$ depends only on $M_{p-1}, \ldots, M_{p-k+1}$ (and not on $M_{p-k}, \ldots, M_I$) since its sum of terms of the form $B_{s-k} \xi^{p-s}$, for $2 \leq s \leq k$, which depend only on $M_j$ with $j \geq p + s - k - 1 \geq p - k + 1$, by (3.10).

Formulas (3.6) and (3.12)-(3.13) together give (3.1). Step 1 is completed.

**Step 2.** We now look at the real part of

$$A_{p-k} := (e^{-\Lambda} A e^\Lambda)_{\text{ord}(p-k)} = A_I|_{\text{ord}(p-k)} + A_{II}|_{\text{ord}(p-k)}$$

$$= A_{p-k}^0 + A_{p-k}^1 + A_{II}|_{\text{ord}(p-k)}, \quad k = 1, \ldots, p - 1.$$
From (1.7)-(1.9), for $|\xi| \geq 2h$, we have
\[
\text{Re} A_{p-k} = \text{Re}(ia_p D_x \lambda_{p-k} \xi^{p-1} + ia_{p-k} \xi^{p-k})
\]
\[
= a_p \xi^{p-1} M_{p-k} \frac{|\xi|^{p-1}}{|\xi|^{p-1}} \frac{\langle x \rangle}{|\xi|^{p-1}} \psi \left( \frac{\langle x \rangle}{|\xi|^{p-1}} \right) \frac{\langle \xi \rangle_h^k}{|\xi|^{p-1}} - \text{Im} a_{p-k} \cdot \xi^{p-k}
\]
\[
\geq \left( \frac{2}{\sqrt{5}} \right) a_p \frac{M_{p-k}}{\langle x \rangle^{p-1}} \frac{\langle \xi \rangle_h^p}{|\xi|^{p-1}} - C \frac{\langle \xi \rangle_h^k}{|\xi|^{p-1}} (1 - \psi)
\]
(3.14)
\[
\geq a_p \psi \cdot (2^{p-1} - \frac{1}{2} M_{p-k} - C) \frac{\langle \xi \rangle_h^p}{|\xi|^{p-1}} - C''
\]
for some $C'' > 0$ since $|\xi| = 2^{p-1} \sqrt{\xi^2 + \xi^2} \geq \frac{2}{\sqrt{5}} \langle \xi \rangle_h$ and $\langle \xi \rangle_h^p/|\xi|$ is bounded on the support of $(1 - \psi)$.

From (3.14), (3.13) and (3.6):
\[
\text{Re} A_{p-k} = \text{Re}(A^0_{p-k}) + \text{Re}(A^1_{p-k}) + \text{Re}(A_{II}|_{\text{ord}(p-k)})
\]
(3.15)
\[
\geq a_p \psi \cdot (2^{p-1} - \frac{1}{2} M_{p-k} - C) \frac{\langle \xi \rangle_h^p}{|\xi|^{p-1}} - C'' - (C_k + C') a_p \frac{\langle \xi \rangle_h^p}{|\xi|^{p-1}},
\]
where the constants $C, C', C'', C_k$ depend only on $M_{p-1}, \ldots, M_{p-k+1}$ and not on $M_{p-k}$.

In particular, for $k = 1$,
\[
\text{Re} A_{p-1} \geq a_p \psi \cdot (2^{p-1} - \frac{1}{2} M_{p-1} - C - C_1 - C'') \frac{\langle \xi \rangle_h^p}{|\xi|^{p-1}} - C''
\]
for some $C'' > 0$.

Since $a_p \geq 0$ by assumption, we can choose $M_{p-1} > 0$ sufficiently large, so that
\[
\text{Re} A_{p-1}(t, x, \xi) \geq -\tilde{C} \quad \forall (t, x, \xi) \in [0, T] \times \mathbb{R} \times \mathbb{R}
\]
for some $\tilde{C} > 0$. Applying the sharp-Gårding Theorem A.1 to $A_{p-1} + \tilde{C}$ we can thus find pseudo-differential operators $Q_{p-1}(t, x, D_x)$ and $R_{p-1}(t, x, D_x)$ with symbols $Q_{p-1}(t, x, \xi) \in S^{p-1}$ and $R_{p-1}(t, x, \xi) \in S^{p-2}$ such that
(3.16) $A_{p-1}(t, x, D_x) = Q_{p-1}(t, x, D_x) + R_{p-1}(t, x, D_x) - \tilde{C}$
\[
\text{Re}(Q_{p-1}(t, x, D_x) v(t, x), v(t, x)) \geq 0 \quad \forall v(t, x) \in C_0^\infty([0, T] \times \mathbb{R}, H^{p-1}(\mathbb{R})
\]
\[
R_{p-1}(t, x, \xi) \sim \psi_1(\xi) D_x A_{p-1}(t, x, \xi) + \sum_{\alpha + \beta \geq 2} \psi_{\alpha, \beta}(\xi) \partial_\xi^\alpha \partial_x^\beta A_{p-1}(t, x, \xi)
\]
with $\psi_1 \in S^{-1}, \psi_{\alpha, \beta} \in S^{(\alpha - \beta)/2}, \psi_1, \psi_{\alpha, \beta} \in \mathbb{R}$.

Therefore, after having applied once the sharp-Gårding Theorem A.1, from (3.3), (3.12) and (3.16) we get:
\[
\sigma(e^{-A} Ae^A) = ia_p \xi^p + \sum_{k=1}^{p-1} A_{p-k} + A_0' = ia_p \xi^p + A_{p-1} + \sum_{k=2}^{p-1} A_{p-k} + A_0'
\]
(3.17)
\[
= ia_p \xi^p + Q_{p-1} + \sum_{k=2}^{p-1} (A_{I}|_{\text{ord}(p-k)} + A_{II}|_{\text{ord}(p-k)} + R_{p-1}|_{\text{ord}(p-k)}) + A_0''
\]
for some $A_0', A_0'' \in S^0$, where $R_{p-1}|_{\text{ord}(p-k)}$ denotes the terms of order $p - k$ of $R_{p-1}$.

Step 2 is completed.
Step 3. In order to reapply Theorem A.1 we now have to investigate the action of the sharp-Gårding Theorem to each term of the form $A_l|_{\text{ord}(p-k)} + A_l|_{\text{ord}(p-k)} + S_{p-k}$, where $S_{p-k}$ denotes terms of order $p-k$ coming from remainders of previous applications of Theorem A.1, for $p-k \geq 3$. In the following substeps we compute and estimate the generic remainder $R(A_l|_{\text{ord}(p-k)}) + R(A_l|_{\text{ord}(p-k)}) + R(S_{p-k})$.

Step 3.1: estimate of $R(A_l|_{\text{ord}(p-k)}) = R(A^0_{p-k}) + R(A^1_{p-k})$.

From the sharp-Gårding Theorem A.1:

\[
R(A^0_{p-k}) = \psi_1 D_x A^0_{p-k} + \sum_{\alpha+\beta \geq 2} \psi_{\alpha,\beta} \partial_x^\alpha D_x^\beta A^0_{p-k}
\]

for real valued $\psi_1 \in S^{-1}$ and $\psi_{\alpha,\beta} \in S^{(\alpha-\beta)/2}$.

We have

\[
\psi_1 D_x A^0_{p-k} = \psi_1 D_x(i \alpha p D_x \lambda_{p-k} \xi^{p-1} + i a_{p-k} \xi^{p-k})
\]

\[
= i \alpha p D_x^2 \lambda_{p-k} (\psi_1 \xi^{p-1}) + i D_x a_{p-k} (\psi_1 \xi^{p-k})
\]

and by (1.7):

\[
|\text{Re}(\psi_1 D_x A^0_{p-k})| \leq |\text{Im} D_x a_{p-k}| \cdot |\psi_1 \xi^{p-k}|
\]

\[
\leq \frac{C \alpha p}{\langle x \rangle^{p-k}} \langle \psi_1 \xi^{p-k} \rangle \leq \frac{C' \alpha p}{\langle x \rangle^{p-k-1}} \langle \xi \rangle^{p-k-1}
\]

\[
\leq a_p \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle^{p-k-1}} \right) \frac{C' \langle \xi \rangle^{p-k-1}}{\langle x \rangle^{p-k-1}} + C''
\]

(3.19)

since $\psi_1 \in S^{-1}$ and $\langle \xi \rangle^{p-k-1}/\langle x \rangle^{p-k-1}$ is bounded on supp$(1 - \psi)$.

Let us now estimate

\[
\sum_{\alpha+\beta \geq 2} \psi_{\alpha,\beta} \partial_x^\alpha D_x^\beta A^0_{p-k} = \sum_{\alpha+\beta \geq 2} \psi_{\alpha,\beta} \partial_x^\alpha D_x^\beta (i \xi^{p-k} a_{p-k} + i \alpha p \xi^{p-1} D_x \lambda_{p-k})
\]

\[
= \sum_{\alpha+\beta \geq 2} \psi'_{\alpha,\beta} \xi^{p-k-\alpha} i D_x^\beta a_{p-k} + \alpha p \sum_{\alpha+\beta \geq 2} i \psi_{\alpha,\beta} \partial_x^\alpha (\xi^{p-1} D_x^\beta \lambda_{p-k})
\]

(3.20)

for $\psi'_{\alpha,\beta} = \psi_{\alpha,\beta}(p-k)(p-k-1) \cdots (p-k-\alpha+1)$.

Note that $\psi'_{\alpha,\beta} \xi^{p-k-\alpha} i D_x^\beta a_{p-k} \in S^{p-k-\alpha-\beta/2}$, so it has to be considered at level $p-k - \alpha-\beta/2$ if $\alpha + \beta$ is even, at level $p-k - \alpha-\beta/2 + 1/2$ if $\alpha + \beta$ is odd, thus at level $p-k + \lfloor \frac{-\alpha-\beta}{2} + \frac{1}{2} \rfloor$. Looking also at its decay as $x \to \infty$, we have by (1.7), for $p-k \geq 3$:

\[
|\text{Re}(\psi'_{\alpha,\beta} \xi^{p-k-\alpha} i D_x^\beta a_{p-k})| \leq \langle \xi \rangle^{p-k-\alpha-\beta/2} \frac{C \alpha p}{\langle x \rangle^{p-k-\lfloor \frac{-\alpha-\beta}{2} + \frac{1}{2} \rfloor}} 
\]

\[
\leq C \alpha p \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle^{p-k-\lfloor \frac{-\alpha-\beta}{2} + \frac{1}{2} \rfloor}} \right) \frac{\langle \xi \rangle^{p-k+\lfloor \frac{-\alpha-\beta}{2} + \frac{1}{2} \rfloor}}{\langle x \rangle^{p-k-\lfloor \frac{-\alpha-\beta}{2} + \frac{1}{2} \rfloor}} + C''
\]

(3.21)

for some $C' > 0$, since

\[
\left[ \frac{b}{2} \right] \geq \left[ \frac{a+b}{2} + \frac{1}{2} \right] \quad \forall a, b \geq 0.
\]

(3.22)
We remark that decay estimates of the form (3.21) are needed until level \( p - k - \frac{\alpha + \beta}{2} \geq \frac{1}{2} \), i.e.

\[
0 \leq \left[ \frac{\beta}{2} \right] \leq p - k - 1, \quad \text{for } p - k \geq 3.
\]

To evaluate the second addend of (3.20) we write:

\[
i\psi_{\alpha,\beta} i\partial_x^\gamma (\xi^{p-1} D_x^{\beta+1} \lambda_{p-k}) = i\psi'_{\alpha,\beta} \xi^{p-1-\alpha} D_x^{\beta+1} \lambda_{p-k}
\]

\[
+ i\psi''_{\alpha,\beta} \sum_{\gamma = 1}^{\alpha} \left( \frac{\alpha}{\gamma} \right) \xi^{p-1-\alpha+\gamma} \partial_\xi D_x^{\beta+1} \lambda_{p-k}
\]

for \( \psi'_{\alpha,\beta}, \psi''_{\alpha,\beta} \in S^{\frac{-\alpha}{2}} \).

From (2.12) we have that \( i\psi''_{\alpha,\beta} \xi^{p-1-\alpha} D_x^{\beta+1} \lambda_{p-k} \in S^{p-1-\frac{\alpha + \beta}{2}} \) and has the “right decay” independently on the assumptions on the coefficients because of

\[
|D_x^{\beta+1} \lambda_{p-k}| \leq \frac{C_{k,\beta}}{\langle x \rangle^{\frac{p-k+\alpha + \beta}{p-1}}},
\]

since \( \beta(p - 1) \geq \left[ -\frac{\alpha + \beta}{2} + \frac{1}{2} \right] \).

To estimate the second addend of (3.24) we write, by (2.24) for \( |\xi| \geq 2h \),

\[
i\psi''_{\alpha,\beta} \left( \frac{\alpha}{\gamma} \right) \xi^{p-1-\alpha+\gamma} \partial_\xi D_x^{\beta+1} \lambda_{p-k} = i\psi''_{\alpha,\beta} \left( \frac{\alpha}{\gamma} \right) \xi^{p-1-\alpha+\gamma} (B_{k,\gamma,\beta+1} + C_{k,\gamma,\beta+1} + D_{k,\gamma,\beta+1})
\]

with

\[
B_{k,\gamma,\beta+1} = M_{p-k} \frac{|\xi|^{p-1}}{\xi^{p-1}} D_x^{\beta} \langle x \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle x \rangle}{\xi^{p-1}} \right) \partial_\xi \langle \xi \rangle_{h}^{-k+1}
\]

\[
C_{k,\gamma,\beta+1} = M_{p-k} \frac{|\xi|^{p-1}}{\xi^{p-1}} \sum_{\beta' = 1}^{\beta} \left( \frac{\beta}{\beta'} \right) D_x^{\beta-\beta'} \langle x \rangle^{-\frac{p-k}{p-1}} D_x^{\beta'} \psi \left( \frac{\langle x \rangle}{\xi^{p-1}} \right) \partial_\xi \langle \xi \rangle_{h}^{-k+1}
\]

\[
D_{k,\gamma,\beta+1} = M_{p-k} \frac{|\xi|^{p-1}}{\xi^{p-1}} \sum_{\beta' = 0}^{\beta} \left( \frac{\beta}{\beta'} \right) D_x^{\beta-\beta'} \langle x \rangle^{-\frac{p-k}{p-1}} \sum_{\gamma' = 1}^{\gamma} \left( \frac{\gamma}{\gamma'} \right) \partial_\xi \langle \xi \rangle_{h}^{-k+1} \cdot \sum_{r_1 + \ldots + r_q = \gamma'} C_{q,r} \sum_{\beta'' = 0}^{\beta'} \left( \frac{\beta'}{\beta''} \right) D_x^{\beta'-\beta''} \psi^{(q)} \left( \frac{\langle x \rangle}{\xi^{p-1}} \right) D_x^{\beta''} \langle \xi \rangle_{h}^{q}.
\]

since \( \omega(s)(\xi/h) = 0 \) for all \( s \geq 1 \) if \( |\xi| \geq 2h \).

Note that, as in (3.25),

\[
\left| D_x^{\beta} \langle x \rangle^{-\frac{p-k}{p-1}} \psi \left( \frac{\langle x \rangle}{\xi^{p-1}} \right) \right| \leq \frac{c}{\langle x \rangle^{\frac{p-k+\alpha + \beta}{p-1}}} \leq \frac{c}{\langle x \rangle^{\frac{p-k+\alpha + \beta}{p-1}}}
\]

for some \( c > 0 \). Moreover, \( i\psi''_{\alpha,\beta}(\gamma) \xi^{p-1-\alpha+\gamma} B_{k,\gamma,\beta+1} \) is of order \( p - k - \frac{\alpha + \beta}{2} \), thus we have, for some \( c' > 0 \):

\[
\left| i\psi''_{\alpha,\beta}(\gamma) \xi^{p-1-\alpha+\gamma} B_{k,\gamma,\beta+1} \langle x, \xi \rangle \right| \leq \frac{c'}{\langle x \rangle^{\frac{p-k+\alpha + \beta}{p-1}}},
\]
On the other hand,
\[
|\psi_{\alpha,\beta}^{m} e^{p-1-\alpha+\gamma}C_{k,\gamma,\beta+1}| \leq c \langle \xi \rangle_{h}^{-\alpha+\beta+\gamma-k+1} \sum_{\beta=1}^{\beta} \langle x \rangle^{-\frac{p-k}{p-1} - \beta + \beta'} \left| \partial_{x}^{\beta'} \psi \left( \frac{\langle x \rangle}{\langle \xi \rangle_{h}^{p-1}} \right) \right|
\]
\[
\leq c \langle \xi \rangle_{h}^{-\frac{p-k}{p-1}} \sum_{\beta'=1}^{\beta} \langle x \rangle^{-\beta + \beta'} \cdot \sum_{r_{1}+\ldots+r_{q}=\beta'} C_{q,r} \left| \psi(q) \right| \left| \partial_{x}^{r_{q}} \frac{\langle x \rangle}{\langle \xi \rangle_{h}^{p-1}} \right| \langle \xi \rangle_{h}^{q(p-1)+\beta'} \chi_{\text{supp } \psi'}
\]
\[
\leq c_{2} \sum_{\beta',\beta''=0}^{\beta} \sum_{\gamma=1}^{\gamma} \sum_{r_{1}+\ldots+r_{q}=\gamma'} \langle \xi \rangle_{h}^{-\alpha+\beta+\gamma-k+1} \langle x \rangle^{-\beta + \beta'} \langle \xi \rangle_{h}^{q(p-1)+\beta'} \chi_{\text{supp } \psi'}
\]
\[
\leq c_{3}
\]
for some \( c, c', c'' > 0 \) since \( \langle \xi \rangle_{h}/\langle x \rangle^{\frac{1}{p-1}} \) is bounded on \( \text{supp } \psi \).

Analogously,
\[
|\psi_{\alpha,\beta}^{m} e^{p-1-\alpha+\gamma}D_{k,\gamma,\beta+1}| \leq c_{1} \sum_{\beta'=0}^{\beta} \sum_{\gamma=1}^{\gamma} \sum_{r_{1}+\ldots+r_{q}=\gamma'} \langle \xi \rangle_{h}^{-\alpha+\beta+\gamma-k+1} \langle x \rangle^{-\beta + \beta'} \leq c_{2}
\]
for some \( c_{1}, c_{2}, c_{3} > 0 \).

Summing up, we have obtained, for the second addend of (3.18), that
\[
|\text{Re} \sum_{\alpha+\beta \geq 2} \psi_{\alpha,\beta} \partial_{\xi}^{\alpha} D_{x}^{\beta} A_{p-k}^{0}| \leq C a_{p} \langle \xi \rangle_{h}^{p-k+1} \langle x \rangle^{-\frac{p-k}{p-1} + \frac{\alpha+\beta + \frac{1}{2}}{2}} \psi + C'
\]
for some \( C, C' > 0 \), because of (3.21), (3.25) and (3.26). Note that only in (3.21) the assumptions (1.7) are used. We have thus proved, looking also at (3.19), that \( R(A_{p-k}^{0}) \) has the “right decay” and, moreover, it depends only on \( M_{p-k} \) and not on \( M_{j} \) for \( j \neq p-k \).

We now estimate the remainder
\[
R(A_{p-k}^{1}) = \sum_{s=2}^{k} R(B_{s-k} e^{p-s})
\]
(3.27)
\[
= \sum_{s=2}^{k} \left[ \psi_{1} D_{x}(B_{s-k} e^{p-s}) + \psi_{\alpha,\beta} \partial_{\xi}^{\alpha} D_{x}^{\beta} (B_{s-k} e^{p-s}) \right]
\]
for \( \psi_{1} \in S^{-1}, \psi_{\alpha,\beta} \in S^{\alpha+\beta} \) and \( B_{s-k} \) defined by (3.10).

We have
\[
\psi_{1} D_{x}(e^{p-s} B_{s-k}) = \psi_{1} e^{p-s} (D_{x} B_{s-k}) \sum_{m=1}^{k} a_{p-s+m} + \psi_{1} e^{p-s} b_{s-k} D_{x} \left( \sum_{m=1}^{k} a_{p-s+m} \right).
\]
From (2.37):

\[
|b_{s-k}| \leq \frac{C_{s-k}}{\langle x \rangle^{p-k+1}} \langle \xi \rangle^{s-k} \leq \frac{C_{s-k}}{\langle x \rangle^{p-k+1}} \langle \xi \rangle^{s-k}
\]

\[
|D_x b_{s-k}| \leq \frac{C'_{s-k}}{\langle x \rangle^{p-k+1}} \langle \xi \rangle^{s-k} \leq \frac{C'_{s-k}}{\langle x \rangle^{p-k+1}} \langle \xi \rangle^{s-k},
\]

dependently of the conditions on the \(x\) for some \(C\).

therefore, for each \(2 \leq s \leq k\),

\[
(3.28) \quad \left| \psi \sum_{s=2}^{k} D_x (B_{s-k} \xi^{s-s}) \right| \leq c a_p \frac{\langle \xi \rangle^{p-k-1}}{\langle x \rangle^{p-k+1}} \leq c a_p \psi \frac{\langle \xi \rangle^{p-k-1}}{\langle x \rangle^{p-k+1}} + c'
\]

for some \(c, c' > 0\), because of (1.6)-(1.9) (with \(\beta = 0\) and not using the assumptions on the decay in \(x\)).

For the second addend of (3.27) we write

\[
\sum_{\alpha + \beta \geq 2} \psi_{\alpha, \beta} \partial_\xi^\alpha D_\xi^\beta (B_{s-k} \xi^{p-s}) = \psi_{\alpha, \beta} \sum_{\alpha = 0}^{a} \left( \frac{\alpha}{\alpha'} \right) (\partial_\xi^\alpha \xi^{p-s}) (\partial_\xi^{\alpha'} D_\xi^\beta B_{s-k})
\]

\[
= \sum_{\alpha + \beta \geq 2} \sum_{\alpha'=0}^{a} \left( \frac{\alpha}{\alpha'} \right) \psi_{\alpha, \beta} \xi^{p-s-\alpha+\alpha'} \sum_{\beta=0}^{\beta} \left( \frac{\beta}{\beta'} \right) (\partial_\xi^{\alpha'} D_\xi^{\beta'} b_{s-k}) D_\xi^{\beta-\beta'} (\sum_{m=1}^{k} a_{p-s+m})
\]

for \(\psi_{\alpha, \beta} \in S_{\frac{a}{2}}^{+}\) and \(\partial_\xi^{\alpha'} D_\xi^{\beta'} b_{s-k} \in S_{s-k-\alpha'}^{-}\), because of (2.37).

Therefore \(\psi_{\alpha, \beta} \xi^{p-s-\alpha+\alpha'} (\partial_\xi^{\alpha'} D_\xi^{\beta'} b_{s-k}) \in S_{p-k-\alpha+\beta}^{+}\) and, by (2.37),

\[
\left| \psi_{\alpha, \beta} \xi^{p-s-\alpha+\alpha'} (\partial_\xi^{\alpha'} D_\xi^{\beta'} b_{s-k}) \right| \leq \frac{C'_{s-k}}{\langle x \rangle^{p-k+1}} \langle \xi \rangle^{p-k-\alpha+\beta} \langle \xi \rangle^{s-k} \leq \frac{C'_{s-k}}{\langle x \rangle^{p-k-1+\frac{\alpha+\beta}{2}}} \langle \xi \rangle^{s-k} \langle \xi \rangle^{p-k+\frac{\alpha+\beta}{2}}
\]

for some \(C'_{s-k} > 0\), since \(\beta' \geq \left[ -\frac{\alpha+\beta}{2} + 1 \right] \).

This, together with (3.28), means that \(R(A_{p-k})\) satisfies the “right decay at the right level”, independently of the conditions on the \(x\)-decay of the coefficients.

**Step 3.2:** estimate of \(R(A_{II} |_{\text{ord}(p-k)})\).

We write

\[
R \left( (i D_\xi^\alpha a_j) (i D_\xi^\beta e^{-\lambda}) (D_\xi^{m+\alpha+\beta} e^\lambda) \xi^{j-m} \right) = \psi_1 D_\xi \left[ (i D_\xi^\alpha a_j) (i D_\xi^\beta e^{-\lambda}) (D_\xi^{m+\alpha+\beta} e^\lambda) \xi^{j-m} \right]
\]

\[
(3.29) + \sum_{\alpha + \beta \geq 2} \psi_{\alpha', \beta'} \partial_\xi^{\alpha'} (i D_\xi^\beta a_j) (i D_\xi^\beta e^{-\lambda}) (D_\xi^{m+\alpha+\beta} e^\lambda) \xi^{j-m}
\]

for \(\psi_1 \in S^{-1}\) and \(\psi_{\alpha', \beta'} \in S_{\frac{\alpha'+\beta'}{2}}^{-}\).

In order to avoid further computations analogous to those already made for the estimate of \(A_{II}\), we make some remarks. When the \(x\)-derivatives fall on \((\partial_\xi^\alpha e^{-\lambda}) (D_\xi^{m+\alpha+\beta} e^\lambda)\), the decay in \(x\) gets better, while the level in \(\xi\) decreases, because of Lemma 2.5. Therefore we still have the “right decay”. When the \(x\)-derivatives fall on \(D_\xi^\beta a_j\) the assumptions (1.7) on the coefficients give a decay in \(\langle x \rangle\) of order \((j - \left[ \frac{p-1}{2} \right])/(p-1)\) in the first addend of (3.29) and of order \((j - \left[ \frac{p+1}{2} \right])/(p-1)\) in the second addend of (3.29); at the same time we have that the level in \(\xi\) decreases of 1 in the first addend of (3.29) and of \(\alpha' - \frac{\alpha'-\beta'}{2} = \frac{\alpha'+\beta'}{2}\) in the second addend of...
(3.29). Therefore the assumptions (1.7) on the coefficients still give the “right decay”, since

\[
-\left\lfloor \frac{\beta + 1}{2} \right\rfloor \geq \left\lfloor -\frac{\beta}{2} - 1 + \frac{1}{2} \right\rfloor
\]

(3.30)

because of (3.22) with \( b = \beta + 1, a = 1 \) and \( b = \beta + \beta', a = \alpha' \) respectively.

Thus, conditions (1.7) still give the “right decay” for \( R(\mathcal{A}_I) \), and hence for \( R(\mathcal{A}_I|_{\text{ord}(p-k)}) \).

**Step 3.3:** estimate of remainders coming from previous applications of the sharp-Gårding Theorem A.1.

To estimate \( S_{p-k} \) and then \( R(S_{p-k}) \) we previously need to make some remarks.

From (3.17) with \( R_{p-1} = R(\mathcal{A}_{p-1}) \) we have

\[
\sigma(e^{-\Lambda} \mathcal{A} e^{\Lambda}) = i a_{p} \xi^{p} + Q_{p-1} + R(\mathcal{A}_{p-1}) + \sum_{k=2}^{p-1} A_{p-k} + A_{0}''
\]

\[
= i a_{p} \xi^{p} + Q_{p-1} + A_{p-2} + R(\mathcal{A}_{p-1})|_{\text{ord}(p-2)} + \sum_{k=3}^{p-1} (A_{p-k} + R(\mathcal{A}_{p-1})|_{\text{ord}(p-k)}) + A_{0}''
\]

From (3.15) for \( k = 2 \) and from the above discussions on the remainders \( R(\mathcal{A}_{0}|_{p-k}), R(\mathcal{A}_{1}|_{p-k}) \) and \( R(\mathcal{A}_{I}|_{\text{ord}(p-k)}) \) for \( k = 1 \), we can now choose \( M_{p-2} > 0 \) sufficiently large so that

\[
\text{Re} \left( A_{p-2} + R(\mathcal{A}_{p-1})|_{\text{ord}(p-2)} \right) (t, x, \xi) \geq -\tilde{C} \quad \forall (t, x, \xi) \in [0, T] \times \mathbb{R} \times \mathbb{R}
\]

for some \( \tilde{C} > 0 \).

Note that \( A_{p-2} \) depends on \( M_{p-1} \) and \( M_{p-2} \) in the sense of (3.15), while \( R(\mathcal{A}_{p-1})|_{\text{ord}(p-2)} \) depends only on the already chosen \( M_{p-1} \). Thus, by the sharp-Gårding Theorem A.1 there exist pseudo-differential operators \( Q_{p-2} \) and \( R_{p-2} \), with symbols in \( S^{p-2} \) and \( S^{p-3} \) respectively, such that

\[
\text{Re}(Q_{p-2} v, v) \geq 0 \quad \forall v(t, \cdot) \in H^{p-2}
\]

\[
A_{p-2} + R(\mathcal{A}_{p-1})|_{\text{ord}(p-2)} = Q_{p-2} + R_{p-2},
\]

with

\[
R_{p-2} = R(\mathcal{A}_{p-2} + R(\mathcal{A}_{p-1})|_{\text{ord}(p-2)}) = R(\mathcal{A}_{p-2}) + R(R(\mathcal{A}_{p-1})|_{\text{ord}(p-2)}),
\]

so that

\[
\sigma(e^{-\Lambda} \mathcal{A} e^{\Lambda}) = i a_{p} \xi^{p} + Q_{p-1} + Q_{p-2} + R(\mathcal{A}_{p-2}) + R(R(\mathcal{A}_{p-1})|_{\text{ord}(p-2)})
\]

\[
+ \sum_{k=3}^{p-1} (A_{p-k} + R(\mathcal{A}_{p-1})|_{\text{ord}(p-k)}) + A_{0}'
\]

\[
= i a_{p} \xi^{p} + Q_{p-1} + Q_{p-2}
\]

\[
+ \left( A_{p-3} + R(\mathcal{A}_{p-1})|_{\text{ord}(p-3)} + R(\mathcal{A}_{p-2})|_{\text{ord}(p-3)} + R^2(\mathcal{A}_{p-1})|_{\text{ord}(p-3)} \right)
\]

\[
+ \sum_{k=4}^{p-1} \left( A_{p-k} + R(\mathcal{A}_{p-1})|_{\text{ord}(p-k)} + R(\mathcal{A}_{p-2})|_{\text{ord}(p-k)} + R^2(\mathcal{A}_{p-1})|_{\text{ord}(p-k)} \right) + A_{0}'.
\]
To proceed analogously for the terms of order $p - 3$, then $p - 4$ and so on up to order 3, we thus need to estimate, for $p - k \geq 3$ and $s \geq 2$:

$$R^s(A_{p-k}) = R^s(A_{p-k}^0) + R^s(A_{p-k}^1) + R^s(A_{II|\text{ord}(p-k)}).$$

The arguments are analogous to those already made for the discussion of $R(A_{p-k}^0)$, $R(A_{p-k}^1)$ and $R(A_{II|\text{ord}(p-k)})$. Indeed, in the remainders of the sharp-Gårding Theorem A.1 we have a first addend with some $\tilde{V}_1 \in S^{-1}$ and where some derivatives $D_x$ appears and a second addend with some $\psi_{\alpha',\beta'} \in S^{'-\beta'}$ and where some derivatives $\partial^\beta_x D_x^\beta'$ appear.

When the $x$-derivatives fall on $\lambda_{p-j}$ the decay in $x$ gets better by (2.13), while the level in $\xi$ decreases, so that we still have the “right decay”.

When the $x$-derivatives fall on the coefficients then the assumptions (1.7) still give the “right decay” since the level in $\xi$ decreases of $\frac{\alpha' + \beta'}{2}$ (for $\alpha' = \beta' = 1$ in the first addend) and because of (3.30).

Therefore, remainders coming from the sharp-Gårding Theorem A.1 always have the “right decay”.

All these computations show that we can apply again and again the sharp-Gårding Theorem A.1 until we find pseudo-differential operators $Q_{p-1}, Q_{p-2}, \ldots, Q_3$ of order $p - 1, p - 2, \ldots, 3$ respectively and all positive definite, such that

$$\sigma(e^{-A}A^p) = i\alpha_p\xi^p + Q_{p-1} + Q_{p-2} + \ldots Q_3 + \sum_{k=p-2}^{p-1} (A_{p-k} + S_{p-k}) + \tilde{A}_0$$

for some $\tilde{A}_0 \in S^0$ and $S_{p-k}$ coming from remainders of the sharp-Gårding theorem.

**Step 4.** We argue similarly as in the previous steps to choose $M_2 > 0$ such that

$$\text{Re}(A_2 + S_2) \geq 0$$

(up to a constant that we can put in $\tilde{A}_0$), but then we apply the Fefferman-Phong inequality (A.3), instead of the sharp-Gårding Theorem A.1, and get that

$$\text{Re}(\langle (A_2 + S_2)v, v \rangle) \geq -c\|v\|_0^2 \quad \forall v(t, \cdot) \in H^2$$

for some $c > 0$, without any remainder.

Finally, we choose $M_1 > 0$ such that

$$\text{Re}(A_1 + S_1) \geq 0,$$

we apply the sharp-Gårding inequality (A.2) for $m = 1$ and we finally get

$$\sigma(e^{-A}A^p) = i\alpha_p\xi^p + \sum_{s=1}^{p-3} Q_{p-s} + (A_2 + S_2) + (A_1 + S_1) + \tilde{A}_0$$

with

$$\text{Re}(Q_{p-s}v, v) \geq 0 \quad \forall v(t, \cdot) \in H^{p-s}, \ s = 1, 2, \ldots, p - 3$$
$$\text{Re}(\langle (A_2 + S_2)v, v \rangle) \geq -c\|v\|_0^2 \quad \forall v(t, \cdot) \in H^2$$
$$\text{Re}(\langle (A_1 + S_1)v, v \rangle) \geq -c\|v\|_0^2 \quad \forall v(t, \cdot) \in H^1.$$
A. Ascanelli, C. Boiti and L. Zanghirati 25

\[ \sigma(e^{-A}A^me^A) = \sum_{s=0}^{p} Q_{p-s}^{n,m} \]

with \( Q_{0}^{n,m} \in S^0 \) and

\[ \text{Re} \langle Q_{p-s}^{n,m}v, v \rangle \geq -C_{n,m} \| v \|_0^2 \quad \forall v(t, \cdot) \in H^{p-s} \quad 1 \leq s \leq p - 1 \]

for some \( C_{n,m} > 0 \).

Since every \( Q \in S^0 \) also satisfies

\[ \text{Re} \langle Qv, v \rangle \geq -c \| v \|_0^2 \quad \forall v \in H^0 \]

for some \( c > 0 \), by Lemma 2.6 we finally have that

\[ \text{Re} \langle A\Lambda v, v \rangle \geq -c \| v \|_0^2 \quad \forall v \in H^\infty \]

for some \( c > 0 \), and hence if \( v \in C([0, T]; H^\infty) \) is a solution of (2.2), by (2.1) with \( A\Lambda \) instead of \( A \) we get that

\[ \frac{d}{dt} \| v \|^2_0 \leq \| f\Lambda \|^2_0 + \| v \|^2_0 - 2 \text{Re} \langle A\Lambda v, v \rangle \leq (2c + 1)(\| f\Lambda \|^2_0 + \| v \|^2_0). \]

By standard arguments we deduce that, for all \( s \in \mathbb{R} \), it holds

\[ \| v(t, \cdot) \|^2_s \leq c'( \| g\Lambda \|^2_s + \int_0^t \| f\Lambda (\tau, \cdot) \|^2_s d\tau ) \quad \forall t \in [0, T], \]

for some \( c' > 0 \).

Since \( e^{\Lambda} \in S^\delta \), for \( u = e^{\Lambda}v \) we finally have, from (3.31) with \( s - \delta \) instead of \( s \):

\[ \| u \|^2_{s-2\delta} \leq c_1 \| v \|^2_{s-\delta} \leq c_2 \left( \| g\Lambda \|^2_{s-\delta} + \int_0^t \| f\Lambda (\tau, \cdot) \|^2_{s-\delta} d\tau \right) \]

\[ \leq c_3 \left( \| g \|^2_s + \int_0^t \| f \|^2_s d\tau \right) \]

for some \( c_1, c_2, c_3 > 0 \).

This proves the existence of a solution \( u \in C([0, T]; H^\infty(\mathbb{R})) \) of (1.2) which satisfies (1.10) for \( \sigma = 2\delta = 2(p - 1)M_{p-1} \).

**Remark 3.1.** We have shown that a loss of derivatives appears in the solution of (1.2). The loss comes from (2.5), more precisely from (2.7). If condition (1.7) for \( \beta = 0 \) and \( j = p - 1 \) is substituted by the slightly stronger condition

\[ | \text{Im} a_{p-1}(t, x) | \leq \frac{Ca_p(t)}{\langle x \rangle^{1+\eta}} \]

for some \( \eta > 0 \), then, by defining

\[ \lambda_{p-1}(x, \xi) = M_{p-1} \omega \left( \frac{\xi}{h} \right) \int_0^x \langle y \rangle^{-1-\eta} \psi \left( \frac{\langle y \rangle}{\langle \xi \rangle_h^{p-1}} \right) dy , \]

(cfr. (2.4)), our method gives well-posedness of (1.2) in Sobolev spaces without any loss of derivatives.
**Remark 3.2.** We remark that, for $\beta = 1$, we can weaken condition (1.7) by

$$| \text{Im} \, D_x a_j(t, x) | \leq \frac{C a_p(t)}{\langle x \rangle^{p-1}} \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \ 3 \leq j \leq p - 1. \quad (3.32)$$

Indeed, a decay of type $\langle x \rangle^{-j + \beta / 2} + (p-1)$ instead of $\langle x \rangle^{-j + \beta}$ is needed only in (3.21) (see also (3.19)), but if $\beta = 1$ we can improve (3.22) by

$$-1 \geq \left[ -\frac{a + 1}{2} + \frac{1}{2} \right] \quad \forall a \geq 1. \quad (3.33)$$

Similarly, in the remainders of Steps 3.2 and 3.3 it’s enough to use (3.32) instead of (1.7) for $\beta = 1$.

**Acknowledgements:** We are really grateful to Dr. Torsten Herrmann who carefully read our paper and pointed out the sufficiency of condition (3.32) for $\beta = 1$.

**Appendix A. Sharp-Gårding and Fefferman-Phong inequalities for pseudo-differential operators**

Let $A(x, D_x)$ be a pseudo-differential operator of order $m$ on $\mathbb{R}$ with symbol $A(x, \xi)$ in the standard class $S^m$ defined by

$$| \partial_\xi^\alpha \partial_x^\beta A(x, \xi) | \leq C_{\alpha, \beta, h} \langle \xi \rangle^{m-\alpha} \quad \forall \alpha, \beta \in \mathbb{N}, \ h \geq 1,$$

for some $C_{\alpha, \beta, h} > 0$.

The following theorem holds (cf. [KG]):

**Theorem A.1 (Sharp-Gårding).** Let $A(x, \xi) \in S^m$ and assume that $\text{Re} \, A(x, \xi) \geq 0$. Then there exist pseudo-differential operators $Q(x, D_x)$ and $R(x, D_x)$ with symbols, respectively, $Q(x, \xi) \in S^m$ and $R(x, \xi) \in S^{m-1}$, such that

$$A(x, D_x) = Q(x, D_x) + R(x, D_x)$$

$$\text{Re} \langle Q(x, D_x) u, u \rangle \geq 0 \quad \forall u \in H^m$$

$$R(x, \xi) \sim \psi_1(\xi) D_x A(x, \xi) + \sum_{\alpha + \beta \geq 2} \psi_{\alpha, \beta}(\xi) \partial_\xi^\alpha \partial_x^\beta A(x, \xi),$$

(A.1)

with $\psi_1, \psi_{\alpha, \beta}$ real valued functions, $\psi_1 \in S^{-1}$ and $\psi_{\alpha, \beta} \in S^{(\alpha-\beta)/2}$.

**Remark A.2.** Theorem A.1 implies the well-known sharp-Gårding inequality

$$\text{Re} \langle A(x, D_x) u, u \rangle \geq -c \| u \|_{S^{-1}}^2,$$

(A.2)

for some $c > 0$.

Moreover, the following theorem holds (cf. [FP]):

**Theorem A.3 (Fefferman-Phong inequality).** Let $A(x, \xi) \in S^m$ with $\text{Re} \, A(x, \xi) \geq 0$. Then

$$\text{Re} \langle A(x, D_x) u, u \rangle \geq -c \| u \|_{S^{-2}}^2$$

(A.3)

for some $c > 0$. 


References


